

Master of Science Thesis

# **Lorentzian Cobordisms, Compact Horizons and the Generic Condition**

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### **Abstract**

We consider the problem of determining which conditions are necessary for cobordisms to admit Lorentzian metrics with certain properties. In particular, we prove a result originally due to Tipler without a smoothness hypothesis necessary in the original proof. In doing this, we prove that compact horizons in a smooth spacetime satisfying the null energy condition are smooth.

We also prove that the "generic condition" is indeed generic in the set of Lorentzian metrics on a given manifold.

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# Introduction

An intriguing question which can be posed in the theory of general relativity is that of *topology change*: Is it possible for a spacelike slice of spacetime at one time to have a different topology than that of a spacelike slice at some other time? One way of making this question precise is through the concept of a *Lorentzian cobordism* (see Definition 1.0.2); that is, a spacetime whose boundary consists of disjoint spacelike submanifolds. The question about whether topology change is possible can then be interpreted as the question of whether physically interesting nontrivial Lorentzian cobordisms exist. For this question to be interesting the cobordism needs to have some compactness property. We will consider both the case when the cobordism is compact, and the case when the cobordism has the weaker property of causal compactness (see Definition 1.0.3). Of course, the most interesting cobordisms from the perspective of topology change are the connected cobordisms, so we will mainly be concerned with those.

## Earlier results

The existence of Lorentzian cobordisms when no geometrical conditions are imposed is essentially a problem of differential topology. It is equivalent to the existence of a cobordism with a vector field with a prescribed direction at the boundary, and the problem of characterizing pairs of manifolds which are cobordant in this sense was considered by Reinhart [29]. We discuss this in Section 1.1.

When geometrical conditions are imposed, it is significantly more difficult to construct Lorentzian cobordisms which are not diffeomorphic to  $S \times [0, 1]$  for some manifold  $S$ . There are two classical results about the non-existence of nontrivial Lorentzian cobordisms under certain hypotheses: In 1967 it was shown by Geroch [16] that nontrivial Lorentzian cobordisms can exist only if they contain closed timelike curves, and in 1977 it was shown by Tipler [32] that nontrivial Lorentzian cobordisms satisfying certain energy conditions cannot exist. It was noted by Borde [4] in 1994 that both of these theorems apply to noncompact but causally compact Lorentzian cobordisms as well, thereby establishing nonexistence of cer-

tain Lorentzian cobordisms between noncompact manifolds when geometrical conditions are imposed.

In contrast to the classical non-existence results, Nardmann [26] has more recently obtained a positive result: Any cobordism, subject to some topological conditions in low dimensions, which has a Lorentzian metric at all can be endowed with a metric satisfying the strong geometrical condition known as the "dominant energy condition". However, they are only "weak Lorentzian cobordisms" in the sense that the boundary is not spacelike.

The paper [33] is concerned with explicit constructions of Lorentzian cobordisms using Morse theory. The related paper [34] constructs nontrivial Lorentzian cobordism-like objects satisfying certain causality and energy conditions. However, the latter paper does not require the cobordisms to satisfy the compactness assumptions used elsewhere, and so the problem considered there is completely different than the problem considered by Geroch and Tipler.

A survey of the problem of topology change can be found in [4].

In proving Tipler's theorem, one works with a compact Cauchy horizon, and the question arises of which regularity a Cauchy horizon has. It was shown in [10] that Cauchy horizons are not necessarily smooth. In [3, Section IV] it was shown that a compact "almost everywhere  $C^2$ " horizon satisfying the null energy condition is everywhere  $C^1$ . In [5, Section 4] it was asked whether *compact* Cauchy horizons are always smooth. This question was answered in the negative in [6], where it was also suggested that an energy condition might be sufficient for making compact horizons smooth.

## Overview of the present work

The main results of this thesis are Theorem 1.3.41 concerning smoothness of compact Cauchy horizons subject to energy conditions, and Theorem 2.6.3 about the genericity of the "generic condition".

It has been known for some time (see for instance [10, Section 1]) that the proof of Tipler's result in [32] makes an implicit smoothness assumption, and hence is not sufficient to prove the theorem as it is stated. This has not been corrected in [4] where Borde proves the result with weaker hypotheses. Tipler's original proof uses arguments from a part of the proof of the Hawking Singularity Theorem [19, p.295-298], and the same implicit assumption can be found there as well. A similar mistake was made in the original proof of the Hawking Area Theorem, and has since been corrected by Chruściel, Delay, Galloway and Howard [8]. Significant work was necessary to fill in the gaps in the proof of the Hawking Area Theorem, and the proof in [8] is technical. Fortunately, their methods may be adapted to the setting of Tipler's theorem and we do so in Section 1.3 in order to present a more

careful proof of the nonexistence of Lorentzian cobordisms which satisfy certain energy conditions.

In the process of giving a complete proof of Tipler's theorem, we prove in Theorem 1.3.41 that compact horizons in a spacetime satisfying the null energy condition are smooth, thereby significantly generalizing the theorem in [3, Section IV], and providing an answer to the question raised in [6, Section 4] of whether compact horizons which satisfy energy conditions are smooth.

In Geroch's original proof [16] of the nonexistence of Lorentzian cobordisms without closed timelike curves, and also in Borde's proof [4] of the result with weaker assumptions, there is a small detail missing. In Section 1.2 we present a proof of the result with this detail added.

In Chapter 2 we prove a result which to our knowledge is completely new. It concerns a certain curvature condition known as the "generic condition". The name implies that this condition should be generic in some sense. We prove that this is indeed the case; the generic condition is generic in the Whitney  $C^4$  topology on the set of all Lorentzian metrics on a given manifold. Previous work in this direction has been performed by Beem and Harris (see [1] and [2]). They considered the generic condition at a point, and derived geometrical consequences of having many non-generic vectors at the same point.

Appendix A contains notation and conventions, while Appendix B contains proofs which are not central to the development of the main results.



# Lorentzian cobordisms

In this chapter, we will discuss the conditions under which topologically nontrivial Lorentzian cobordisms can exist. In Section 1.1 we briefly discuss this question without any geometrical constraints. We continue by addressing in Section 1.2 how causality conditions affect the existence of nontrivial Lorentzian cobordisms, and conclude in Section 1.3 with a discussion of the relation between energy conditions and nontrivial Lorentzian cobordisms. Notation and conventions can be found in Appendix A.

We will need several different notions of Lorentzian cobordisms. Since the word "cobordism" generally refers to a compact space, we will define the notion of a "Lorentzian pseudocobordism":

**Definition 1.0.1.** Let  $S_1$  and  $S_2$  be manifolds of dimension  $n$  without boundary. A *Lorentzian pseudocobordism* between  $S_1$  and  $S_2$  is a Lorentzian  $(n + 1)$ -manifold  $M$ , the boundary of which is the disjoint union  $S_1 \sqcup S_2$ , such that  $S_1$  and  $S_2$  are spacelike, and  $M$  admits a nonvanishing timelike vector field which is inward-directed on  $S_1$  and outward-directed on  $S_2$ .

The classical notion of a Lorentzian cobordism is the following.

**Definition 1.0.2.** A Lorentzian pseudocobordism  $M$  between  $S_1$  and  $S_2$  is a *compact Lorentzian cobordism* (or simply *Lorentzian cobordism*) if  $M$  is compact.

It turns out, as was noted by Borde (see [4]), that many of the theorems about Lorentzian cobordisms continue to hold when the property of compactness is replaced by the property of "causal compactness". We will call the resulting object a "causally compact Lorentzian cobordism":

**Definition 1.0.3.** A spacetime  $M$  is *causally compact* if  $\overline{I(p)}$  is compact for each  $p \in M$ .

Causal compactness captures the concept of "compact in time", while allowing the spacetime to be non-compact in the spatial directions.

**Definition 1.0.4.** A Lorentzian pseudocobordism  $M$  between  $S_1$  and  $S_2$  is called a *causally compact Lorentzian pseudocobordism* if  $M$  is causally compact.

**Remark 1.0.5.** Of course, we immediately see that every (compact) Lorentzian cobordism is also a causally compact Lorentzian pseudocobordism.

Finally, we define the notion of a "weak Lorentzian cobordism", following [26]. The difference between a Lorentzian cobordism and a weak Lorentzian cobordism is that the boundary of a weak Lorentzian cobordism does not need to be space-like. However, we still require that there is a timelike vector field transverse to the boundary.

**Definition 1.0.6.** Let  $S_1$  and  $S_2$  be manifolds of dimension  $n$  without boundary. A *weak Lorentzian pseudocobordism* between  $S_1$  and  $S_2$  is a Lorentzian  $(n + 1)$ -manifold  $M$ , the boundary of which is the disjoint union  $S_1 \sqcup S_2$ , such that  $M$  admits a nonvanishing timelike vector field which is inward-directed on  $S_1$  and outward-directed on  $S_2$ .

**Definition 1.0.7.** A weak Lorentzian pseudocobordism is a *weak Lorentzian cobordism* if it is compact.

Each weak Lorentzian cobordism may be transformed into a Lorentzian cobordism by a homotopy of the metric making the light cones narrower. The difference between the notions arises when one considers Lorentzian cobordisms satisfying energy conditions. We will discuss this further at the end of Section 1.3.

**Definition 1.0.8.** A Lorentzian pseudocobordism  $M$  between  $S_1$  and  $S_2$  is *topologically trivial* if it is diffeomorphic to  $S_1 \times [0, 1]$ .

## 1.1 Existence of Lorentzian cobordisms

Suppose that we are given two compact manifolds  $S_1$  and  $S_2$  without boundary. A complete characterization of when there is a Lorentzian cobordism between them follows from a result due to Reinhart [29]. The existence result we present in this section is attributed to both Reinhart and Misner (in dimension 3) in [16]. We prove a lemma to explicitly translate Reinhart's result to our setting.

**Lemma 1.1.1.** *Suppose that  $M$  is a compact  $n$ -manifold with boundary  $S_1 \sqcup S_2$ . Then there is a Lorentzian metric  $g$  which makes  $M$  a Lorentzian cobordism between  $S_1$  and  $S_2$  if and only if there is a nonvanishing vector field  $V$  on  $M$  which is inward-directed on  $S_1$  and outward-directed on  $S_2$ .*

*Proof.* The existence of such a vector field  $V$  is included in the definition of a Lorentzian cobordism, so one implication is trivial.

For the converse, suppose that such a vector field  $V$  is given. Choose an arbitrary Riemannian metric  $\sigma$  on  $M$ . (This can be done by lifting metrics from  $\mathbb{R}^n$  to coordinate neighborhoods of  $M$  and patching them together using a partition of unity. The result is a Riemannian metric since the space of Riemannian metrics is convex.) We may normalize  $V$  so that it has unit length in  $\sigma$ . Define a tensor  $g$  on  $M$  by

$$g(X, Y) = \sigma(X, Y) - C\sigma(X, V)\sigma(Y, V)$$

where  $C$  is a constant to be determined. Note that if  $C > 1$  then  $g$  is a Lorentzian metric on  $M$ . This is seen by considering a  $\sigma$ -orthonormal basis containing  $V_p$  for the tangent space  $T_p M$  at some point  $p$ . Let  $i \in \{0, 1\}$ . For each  $p \in S_i$ , there is some value of  $C > 1$  making  $T_p S_i$  spacelike, since  $V$  is by hypothesis transverse to  $T_p S_i$ . All smaller values of  $C$  also make  $T_p S_i$  spacelike. Moreover, If some value of  $C$  makes  $T_p S_i$  spacelike for some point  $p \in S_i$ , then  $T_q S_i$  is also spacelike for all  $q$  in a neighborhood of  $p$ . Since  $S_1 \cup S_2$  is compact a finite number of such neighborhoods with corresponding values of  $C_i$  suffice to cover them, and hence  $C = \min C_i > 1$  makes  $S_1$  and  $S_2$  spacelike hypersurfaces in the Lorentzian manifold-with-boundary  $M$ . This completes the proof.  $\square$

**Theorem 1.1.2.** *Let  $S_1$  and  $S_2$  be compact manifolds. Then there is a Lorentzian cobordism between them if and only if they have the same Stiefel-Whitney numbers and Euler numbers.*

*Proof.* By [29, Theorem 1] the hypotheses about Stiefel-Whitney numbers and Euler numbers are equivalent to there being a cobordism between  $S_1$  and  $S_2$  with a nonvanishing vector field which is inward-directed on  $S_1$  and outward-directed on  $S_2$ . By the previous lemma, this is equivalent to there being a Lorentzian cobordism between  $S_1$  and  $S_2$ .  $\square$

In particular we have the following result in dimension 4, due to the fact that any two 3-manifolds are cobordant (see Theorem A.1.9 and Theorem A.1.10) and have the same Euler numbers (as can be seen by Poincaré duality, see [18, Corollary 3.37]).

**Corollary 1.1.3.** *Let  $S_1$  and  $S_2$  be compact manifolds of dimension 3. Then there is a Lorentzian cobordism between them.*

These results do not answer the question about existence of causally compact Lorentzian pseudocobordisms. We leave the following question unanswered.

**Problem 1.1.4.** Given two manifolds  $S_1$  and  $S_2$ , and a (not necessarily compact) manifold  $M$  with boundary  $S_1 \sqcup S_2$ , does there exist a metric on  $M$  which makes  $M$  a causally compact Lorentzian pseudocobordism between  $S_1$  and  $S_2$ ?

## 1.2 Lorentzian cobordisms and causality

The classical result about the relation between Lorentzian cobordisms and causality conditions is a theorem due to Geroch [16, Theorem 2]. It states that a topologically nontrivial Lorentzian cobordism cannot satisfy the chronology condition. It has been noted previously (see [26, Footnote 2, p. 28]) that the theorem is false as it is stated by Geroch. These issues may be resolved by requiring, as we have done in the definition of a Lorentzian cobordism, that the decomposition  $\partial M = S_1 \sqcup S_2$  is such that an inward-directed timelike vector field on  $S_1$  is future-directed, while an inward-directed timelike vector field on  $S_2$  is past-directed. However, there is a detail missing, both from Geroch's original proof and from the proof given by Borde [4, Theorem 1]. We will first provide a complete proof of the theorem, using the weaker compactness condition of causal compactness suggested by Borde [4], and then discuss which detail is missing from the earlier proofs.

**Theorem 1.2.1.** *Let  $n \geq 1$ , let  $S_1, S_2$  be  $n$ -manifolds without boundary (not necessarily compact, nor necessarily a priori connected). Let  $(M, g)$  be a connected causally compact weak Lorentzian pseudocobordism between  $S_1$  and  $S_2$  which satisfies the chronology condition. Then there is a diffeomorphism  $\varphi: S_1 \times [0, 1] \rightarrow M$  such that the submanifold  $\varphi(\{x\} \times [0, 1])$  is timelike for every  $x \in S_1$ ; in particular  $M$  is topologically trivial, and  $S_1$  and  $S_2$  are diffeomorphic.*

*Proof.* Let  $V$  denote the timelike vector field which is assumed to exist on  $M$ . Our first goal is to prove that each flow line of  $V$  intersects  $S_1$  precisely once and  $S_2$  precisely once. Consider an inextendible future-directed flow line  $\gamma$ . Either  $\gamma$  has a future endpoint, or it does not. If it does not, then  $\gamma$  restricts to a timelike curve  $[0, \infty) \rightarrow M$ . Since it is timelike, its image is contained in  $\overline{I^+(\gamma(0))}$ , which is compact by causal compactness of  $M$ . Hence the curve has some limit point  $x$ , in the sense that it intersects each sufficiently small neighborhood of  $x$  an infinite number of times. By modifying the curve slightly one may obtain a timelike curve which passes through  $x$  twice with the same tangent vector, and this contradicts the hypothesis that  $M$  is chronological. Hence  $\gamma$  has a future endpoint.

The future endpoint of  $\gamma$  belongs either to  $S_1$ , to the interior of  $M$ , or to  $S_2$ . The first case is impossible since  $V$  is inward-directed on  $S_1$ . In the second case  $V$  must be zero at the endpoint, which contradicts the hypothesis that  $V$  is nonzero at each point. Hence the future endpoint of  $\gamma$  belongs to  $S_2$ . By similar arguments,  $\gamma$  has a past endpoint which belongs to  $S_1$ .

We have noted that each flow line of  $V$  intersects  $S_1$  and  $S_2$  at precisely one point each. Let  $L_p$  denote the flow line through a point  $p$ . Let  $U_2$  be an open neighborhood of  $S_2$  such that no integral curve passes the boundary of  $U_2$  twice. Let  $\eta_2$  be a function on  $M$  which is 1 outside of  $U_2$ , and 0 precisely on  $S_2$ . Consider the vector field  $\eta_2 V$ . The flow lines of this vector field on  $M \setminus S_2$  are the same as those of  $V$ . Since each point in  $M \setminus S_2$  belongs to some such flow line and hence can be reached by flowing from  $S_1$ , and each point of  $S_1$  can be flowed an arbitrary time along  $\eta_2 V$  without leaving  $M \setminus S_2$ , the vector field  $\eta_2 V$  induces a diffeomorphism  $\psi_1: S_1 \times [0, \infty) \rightarrow M \setminus S_2$ . By choosing an orientation-preserving diffeomorphism  $\rho: [0, \infty) \rightarrow [0, 1)$  we get a diffeomorphism  $\phi_1: S_1 \times [0, 1) \rightarrow M \setminus S_2$  defined by  $\phi_1(p, t) = \psi_1(p, \rho(t))$ .

Similarly, choosing a neighborhood  $U_1$  of  $S_1$  and a smooth function  $\eta_1$  which is 1 outside of  $U_1$  and 0 on  $S_1$  we get a diffeomorphism  $\phi_2: S_2 \times (0, 1] \rightarrow M \setminus S_1$ .

Note that the flow lines of  $\phi_1$  and  $\phi_2$  are the same as for the original vector field  $V$ , so in particular it holds for each  $p \in S_1$  that  $\phi_1(\{p\} \times (0, 1)) = \phi_2(\{q\} \times (0, 1))$  for a unique  $q \in S_2$ . We introduce a function mapping each point  $p$  to the corresponding point  $q$ : Define the function  $q: S_1 \rightarrow S_2$  by

$$\phi_1(\{p\} \times (0, 1)) = \phi_2(\{q(p)\} \times (0, 1)).$$

A priori, the only thing we know about the function  $q$  is that it is bijective. We will now show that it is in fact a diffeomorphism. By composing  $\phi_1$  and  $\phi_2^{-1}$  we get a diffeomorphism  $\beta: S_1 \times (0, 1) \rightarrow S_2 \times (0, 1)$ . This diffeomorphism is of the form

$$(p, t) \mapsto (q(p), \alpha(p, t))$$

for some function  $\alpha$ . Precomposing  $\beta$  with the inclusion  $i: S_1 \hookrightarrow S_1 \times (0, 1)$  given by  $p \mapsto (p, 1/2)$  and postcomposing it with the projection  $\pi: S_2 \times (0, 1) \rightarrow S_2$  we see that

$$\pi \circ \beta \circ i = q.$$

Since all three of these maps are smooth, so is  $q$ . Similarly, using an inclusion  $i: S_2 \hookrightarrow S_2 \times (0, 1)$  and the projection  $\pi: S_1 \times (0, 1) \rightarrow S_1$  we see that

$$\pi \circ \beta^{-1} \circ i = q^{-1}$$

is also smooth. This shows that  $q$  is a diffeomorphism. In particular,  $S_1$  and  $S_2$  are diffeomorphic.

Let  $\hat{\beta}$  denote the map  $(q^{-1}, \text{id}) \circ \beta$ , i.e. the map  $\hat{\beta}(p, t) = (p, \alpha(p, t))$ . This map is a composition of diffeomorphisms, and hence itself a diffeomorphism. Consider the quotient space of  $S_1 \times [0, 1) \sqcup S_1 \times (0, 1]$  defined by identifying each

$(p, t) \in S_1 \times (0, 1)$  with  $\hat{\beta}(p, t)$ . This space is a smooth manifold diffeomorphic to  $S_1 \times [0, 1]$ . Define the map  $\phi: S_1 \times [0, 1] \sqcup S_1 \times (0, 1] \rightarrow M$  by

$$\phi(p, t) = \begin{cases} \phi_1(p, t) & \text{if } (p, t) \in S_1 \times [0, 1], \\ \phi_2(q(p), t) & \text{if } (p, t) \in S_1 \times (0, 1]. \end{cases}$$

This map descends to a map  $S_1 \times [0, 1] \rightarrow M$  since if  $(p, t) \in S_1 \times (0, 1)$  then

$$(\phi_2 \circ (q, \text{id}) \circ \hat{\beta})(p, t) = (\phi_2 \circ (q, \text{id}) \circ (q^{-1}, \text{id}) \circ \beta)(p, t) = (\phi_2 \circ \beta)(p, t) = \phi_1(p, t).$$

This map is bijective and agrees locally with the diffeomorphisms onto their images  $\phi_1$  and  $\phi_2 \circ (q, \text{id})$ , and so is a diffeomorphism. We have then constructed a diffeomorphism  $S_1 \times [0, 1] \rightarrow M$ . Moreover, by construction it holds for each  $p \in S_1$  that  $\varphi(\{p\} \times [0, 1])$  is an integral curve of the timelike vector field  $V$ , so it is a timelike curve. This completes the proof.  $\square$

**Remark 1.2.2.** Geroch's original proof [16, Theorem 2] of this statement, after it has been shown that each integral curve intersects both  $S_1$  and  $S_2$ , proceeds by constructing a Morse function  $\varphi$  using parametrizations of the integral curves. However, it is not shown that this function is continuous, and this fact is nontrivial. Borde's proof [4, Theorem 1] does not construct a diffeomorphism explicitly, but rather implies that such a diffeomorphism can be constructed from a suitable parametrization of the integral curves. However, one would need to show that such a parametrization can be chosen smoothly. In our proof, this is done by using the cut-off functions  $\eta_1$  and  $\eta_2$  to "normalize" the parametrizations of the integral curves to have infinite length.

In a sense, Geroch's theorem completely answers the question of which causality conditions can hold for a topologically nontrivial causally compact Lorentzian pseudocobordism. Such a space cannot be chronological, but is on the other hand always non-totally vicious since points on the boundaries cannot be part of any closed timelike curve. This means that we know where on the causal ladder described in Section A.2.2 a topologically non-trivial pseudocobordism belongs. The remaining question then concerns causality of topologically trivial pseudocobordisms.

**Example 1.2.3** (Chronology of trivial cobordisms). Figure 1.1 shows a topologically trivial cobordism which is not chronological. Figure 1.3 provides a further illustration of some of the timelike curves in that cobordism. By a similar construction, shown in Figure 1.4, one may obtain a metric on  $S^1 \times [0, 1]$  for which the inward-directed normal vector field is future-directed at both boundary components.

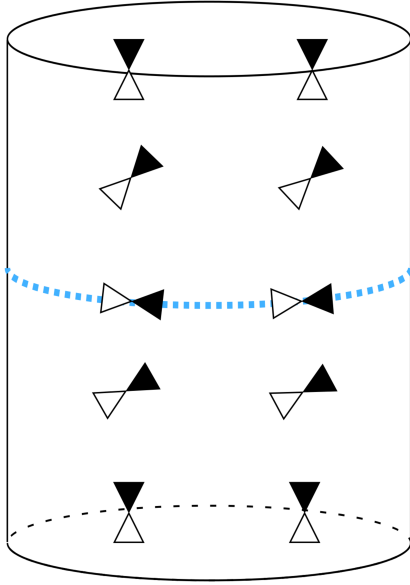


Figure 1.1: A topologically trivial non-chronological cobordism  $S^1 \times [0, 1]$ . It contains a closed timelike curve shown in blue.

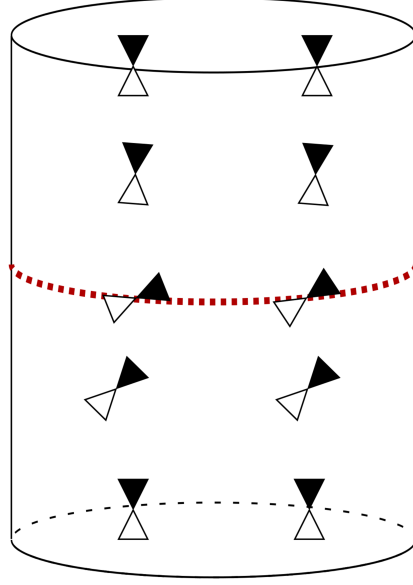


Figure 1.2: A topologically trivial chronological cobordism  $S^1 \times [0, 1]$  which is not causal. It contains a closed lightlike curve shown in red.

**Example 1.2.4** (Causality of trivial cobordisms). Figure 1.2 shows a chronological but non-causal topologically trivial Lorentzian cobordism.

By using the theory of Cauchy horizons, which will be described in greater detail in Section 1.3, one may show the following. We suggest that the reader refers to Section 1.3 for the terminology used in the proof.

**Proposition 1.2.5.** *Let  $M$  be a causally compact Lorentzian pseudocobordism between  $S_1$  and  $S_2$ . Suppose that  $M$  is strongly causal. Then  $M$  is globally hyperbolic.*

*Proof.* Choose the time orientation so that an inward-directed vector field at  $S_1$  is future-directed. Consider the Cauchy horizon  $H^+(S_1)$ . If it is empty, then  $M$  is globally hyperbolic and we are done. If not, choose some point  $p \in H^+(S_1)$  and consider the intersection  $H^+(S_1) \cap \overline{I^-(p)}$ . Choose a generator  $\gamma$  through  $p$ . Since  $J^-(p) \subseteq \overline{I^-(p)}$  by [7, Corollary 2.4.19] we know that  $\text{im } \gamma$  is contained in the compact set  $\overline{I^-(p)}$ . This means that the set  $\{\gamma(k)\}_{k \in \mathbb{N}}$  must have a point of accumulation  $q$  in the compact set  $\overline{I^-(p)}$ . Each sufficiently small neighborhood of  $q$  is then intersected an infinite number of times by the causal curve  $\gamma$ , contradicting the hypothesis that  $M$  is strongly causal. Hence  $H^+(S_1)$  must be empty and we conclude that  $M$  is globally hyperbolic.  $\square$

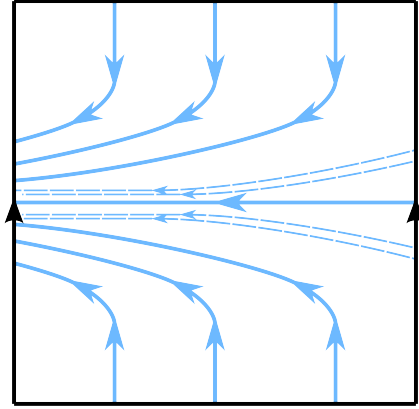


Figure 1.3: Some timelike curves in the cobordism  $S^1 \times [0, 1]$  shown in Figure 1.1. The black arrows indicate that the vertical edges should be identified.

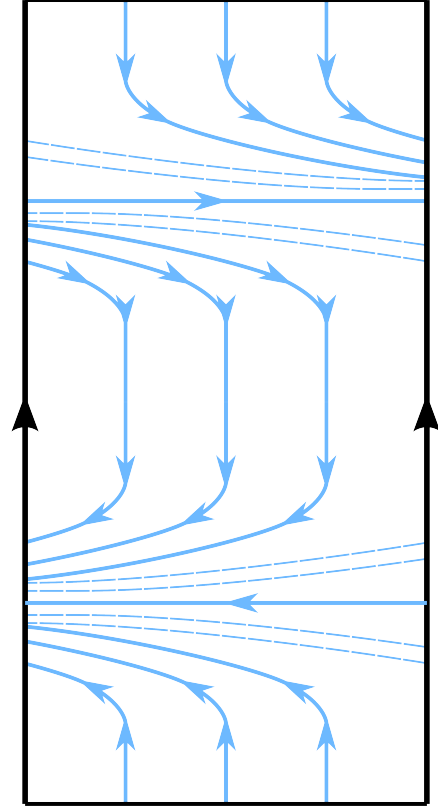


Figure 1.4: Timelike curves for a metric on  $S^1 \times [0, 1]$  which reverses the time orientation.

We leave the remaining possibility as an open problem.

**Problem 1.2.6.** Does there exist a causally compact Lorentzian pseudocobordism which is causal but not strongly causal?

### 1.3 Lorentzian cobordisms and energy conditions

In Section 1.2 we saw a theorem of Geroch [16, Theorem 2] stating that a topologically nontrivial Lorentzian cobordism cannot satisfy the chronology condition. A result from 1977 by Tipler [32, Theorems 3 and 4] further implies that a nontrivial Lorentzian cobordism cannot satisfy certain energy conditions. Unfortunately Tipler's original proof, the methods of which are also used in [19, p.295-298] for proving Hawking's singularity theorem, is flawed in that it is implicitly assumed that a certain Cauchy horizon is  $C^2$ . In this section, we will prove Tipler's result



(Theorem 1.3.46) by a method which does not require such an assumption. The machinery we use is from [8], where similar arguments were used to prove Hawking's Area Theorem. Our proof of Tipler's theorem also yields a smoothness result for compact Cauchy horizons subject to energy conditions. See Theorem 1.3.41.

### 1.3.1 $C^2$ null hypersurfaces

#### 1.3.1.1 The null Weingarten map

In this section we summarize properties of  $C^2$  null hypersurfaces which we will need later. For details, see [15], [22], [14, Section II.1] or [8, Appendix A].

A null hypersurface  $\mathcal{H}$  in a spacetime  $M$  is characterized by the fact that each tangent space  $T_p\mathcal{H}$  contains a unique (up to scaling) null vector  $K_p$ . It holds that the tangent space  $T_p\mathcal{H}$  consists of those vectors of  $T_pM$  which are orthogonal to  $K_p$ . This means that any normal vector field  $K$  of  $\mathcal{H}$  consists entirely of null vectors. We will call the integral curves of these vector fields *generators* of  $\mathcal{H}$ . By [15, Proposition 3.1] these generators (when given a suitable parametrization) are geodesics. By straightforward computations it holds that

$$\langle X, Y \rangle = \langle X', Y' \rangle \quad \text{and} \quad \langle \nabla_X K, Y \rangle = \langle \nabla_{X'} K, Y' \rangle$$

whenever  $X, Y \in T_p\mathcal{H}$  and  $X - X' = \lambda_1 K$  and  $Y - Y' = \lambda_2 K$  for some real numbers  $\lambda_1, \lambda_2$ . Inspired by this, we work instead with the quotient space  $T_p\mathcal{H}/\mathbb{R}K$ . To simplify notation we will denote this space by  $T_p\mathcal{H}/K$ , and the corresponding vector bundle by  $T\mathcal{H}/K$ . This quotient is independent of the particular choice of null vector field  $K$ , since all such vector fields differ only by scaling. We now define the *null Weingarten map* of  $\mathcal{H}$  with respect to  $K$  by

$$b_K: T\mathcal{H}/K \rightarrow T\mathcal{H}/K,$$

$$b_K(\bar{X}) = \overline{\nabla_X K}.$$

This map is not independent of the particular choice of null vector field  $K$ . However, if  $f$  is a smooth nowhere vanishing function then  $b_{fK} = f b_K$  since  $K$  is null. Note that if  $\mathcal{H}$  is  $C^2$ , then  $K$  can be chosen  $C^1$  so that  $b_K$  is continuous. Since all our spacetimes are time-oriented we may restrict attention to future-directed null vector fields  $K$ . This means that we can associate to each null hypersurface a family of null Weingarten maps which differ only by positive scaling. Since  $K$  is null the spacetime metric induces an inner product on  $T\mathcal{H}/K$ . Using this inner product, we may define the *null second fundamental form* of  $\mathcal{H}$  with respect to  $K$  by

$$B_K(\bar{X}, \bar{Y}) = \langle b_K(\bar{X}), \bar{Y} \rangle.$$

A straightforward computation shows that  $B_K$  is symmetric. We will need the following theorem, a proof of which can be found in [22, Theorem 30].

**Theorem 1.3.1.** *Let  $\mathcal{H}$  be a smooth null hypersurface in a spacetime  $M$ . Then the null second fundamental form of  $\mathcal{H}$  is identically zero if and only if  $\mathcal{H}$  is a totally geodesic submanifold of  $M$ .*

**Remark 1.3.2.** The theorem as stated in [22, Theorem 30] applies to null submanifolds in general, regardless of codimension, and so requires the submanifold to be "irrotational". This condition is automatically satisfied for null hypersurfaces.

Finally, we define the *null mean curvature*  $\theta_K$  of a null hypersurface with respect to a null vector field  $K$  as the trace of the null Weingarten map:

$$\theta_K = \text{tr } b_K.$$

Recall that if  $K' = \lambda K$  is another null vector field then  $b_{K'} = \lambda b_K$ . Hence  $\theta_{K'} = \lambda \theta_K$ . This means that the sign of  $\theta_K$  is independent of the particular future-directed null vector field  $K$  used to compute  $\theta_K$ . We will sometimes omit the vector field  $K$  from the notation.

Recall that the integral curves of a null vector field  $K$  are reparametrizations of geodesics. If  $K$  is chosen to agree with  $\dot{\gamma}$  of a geodesic segment  $\gamma$  with affine parameter  $s$ , and  $b(s)$  is the family of null Weingarten maps with respect to  $K$  along  $\gamma$ , then

$$b' + b^2 + \tilde{R} = 0. \quad (1.1)$$

Here  $'$  denotes derivative along  $\gamma$ , and  $\tilde{R}$  denotes the fiberwise endomorphism  $\tilde{R}: T\mathcal{H}/K \rightarrow T\mathcal{H}/K$  defined from the Riemann curvature tensor  $R$  by letting  $\tilde{R}(\bar{X}) = \overline{R(X, \dot{\gamma})\dot{\gamma}}$ . Note that it is not obvious that the derivative  $b'$  exists, since  $b$  is a priori only continuous. A proof of the fact that the derivative does exist and satisfies equation (1.1) can be found in [8, Proposition A.1].

From equation (1.1) one can derive the "Raychaudhuri equation". In particular, one may derive a certain differential inequality for the null mean curvature. Let  $b$  be the null Weingarten map of a  $C^2$  null hypersurface with respect to a future-directed null vector field  $K$  (scaled to give an affine parametrization of an integral curve), and let  $\theta$  denote the trace of  $b$ . Let  $S = b - \frac{\theta}{n-2} \text{id}$ . Then the trace of  $b^2$  is  $\text{tr } b^2 = \theta^2/(n-2) + \text{tr}(S^2)$  so taking the trace of equation (1.1) yields

$$\theta' + \frac{\theta^2}{n-2} + \text{tr}(S^2) + \text{Ric}(K, K) = 0. \quad (1.2)$$

Since  $b$  and  $\text{id}$  are self-adjoint, so is  $S$ . Hence  $\text{tr}(S^2) \geq 0$  so

$$\theta' + \frac{\theta^2}{n-2} + \text{Ric}(K, K) \leq 0. \quad (1.3)$$

This is the differential inequality we will use later.

### 1.3.1.2 Generator flow on $C^2$ null hypersurfaces

A null vector field on a  $C^2$  null hypersurface gives rise to a family of diffeomorphisms which flow points along the vector field. The integral curves of such a vector field are called generators, and we will refer to such a flow as a *generator flow*. The generator flow for time  $t$  will typically be denoted  $\beta_t$ . Given a Riemannian metric  $\sigma$  with volume form  $\omega$ , the *Jacobian determinant*  $J(\beta_t)$  with respect to  $\sigma$  is the real valued function defined by  $(\beta_t)^*\omega = J(\beta_t)\omega$ . In this section, we will show that Jacobian determinant of a generator flow with respect to a certain family of Riemannian metrics is related to the null mean curvature of the hypersurface. We choose to work with a past-directed vector field  $T$  since this is the case in which we will apply the lemma.

**Lemma 1.3.3.** *Let  $\mathcal{H}$  be a  $C^2$  null hypersurface in a spacetime  $(M, g)$  of dimension  $n+1$ . Let  $V$  be an arbitrary unit timelike vector field on  $M$ , and define a Riemannian metric  $\sigma$  on  $M$  by*

$$\sigma(X, Y) = g(X, Y) + 2g(X, V)g(Y, V).$$

*Let  $T$  denote the unique past-directed lightlike  $\sigma$ -unit vector field on  $\mathcal{H}$  and let  $\omega$  denote the  $\sigma$ -volume form induced on  $\mathcal{H}$ . Let  $\theta$  denote the null mean curvature of  $\mathcal{H}$  with respect to the future-directed null vector field  $-T$ . Then the Lie derivative of  $\omega$  with respect to  $T$  is  $\mathcal{L}_T\omega = -\theta\omega$ .*

*Proof.* Choose some point  $p \in \mathcal{H}$  at which to evaluate  $\mathcal{L}_T\omega$ . Let  $e_1, e_2, \dots, e_n$  be a  $g$ -orthogonal basis for  $T_p\mathcal{H}$  such that

- $e_1 = T_p$ ,
- $g(e_i, V) = 0$  for  $i = 2, 3, \dots, n$ ,
- $g(e_i, e_i) = 1$  for  $i = 2, 3, \dots, n$ .

Recall that integral curves of null vector fields on null hypersurfaces are geodesic segments. Let  $\gamma$  be a segment of the integral curve of  $T$  through  $p$  with an affine parametrization. Extend the basis  $e_1, \dots, e_n$  along  $\gamma$  by letting

- $e_1 = T$ ,
- $\nabla_{e_1} e_i = 0$  for  $i = 2, 3, \dots, n$ .

Here  $\nabla$  denotes covariant derivative with respect to  $g$ . Note that we do not yet know that  $(e_i)_{i=1}^n$  is a frame for  $\mathcal{H}$ , since we first need to show that the  $e_i$  are tangent to  $\mathcal{H}$ . This will follow from Claim II below.

**Claim I:**  $g(e_1, V) = 1/\sqrt{2}$  on all of  $\gamma$

Since  $e_1$  is  $g$ -lightlike and  $\sigma$ -unit

$$1 = \sigma(e_1, e_1) = g(e_1, e_1) + 2(g(e_1, V))^2 = 2(g(e_1, V))^2$$

proving the claim.

**Claim II:**  $g(e_1, e_i) = 0$  on all of  $\gamma$

By choice of the basis at  $p$ , we know that  $g(e_1, e_i) = 0$  at  $p$ . It holds on  $\gamma$  that

$$e_1(g(e_1, e_i)) = g(\nabla_{e_1} e_1, e_i) + g(e_1, \nabla_{e_1} e_i) = g(\nabla_{e_1} e_1, e_i)$$

since  $\nabla_{e_1} e_i = 0$ . Since the integral curve of  $T$  is a reparametrization of the geodesic  $\gamma$  it holds that  $\nabla_{e_1} e_1 = \lambda e_1$  for some function  $\lambda$  along  $\gamma$ . Hence

$$e_1(g(e_1, e_i)) = \lambda g(e_1, e_i).$$

This can be seen as an ordinary differential equation for  $g(e_1, e_i)$  along  $\gamma$ , and together with the initial value  $g(e_1, e_i) = 0$  at  $p$ , the uniqueness theorem of ordinary differential equations tells us that  $g(e_1, e_i) = 0$  on all of  $\gamma$ .

**Claim III:**  $g(e_i, e_j) = \delta_{ij}$  for  $i, j = 2, 3, \dots, n$  on all of  $\gamma$

By choice of the basis at  $p$ , the identity holds at  $p$ . By choice of the extensions  $e_i$  and  $e_j$

$$e_1 g(e_i, e_j) = g(\nabla_{e_1} e_i, e_j) + g(e_i, \nabla_{e_1} e_j) = g(0, e_j) + g(e_i, 0) = 0.$$

Viewing this as an initial value problem tells us that  $g(e_i, e_j) = \delta_{ij}$  on all of  $\gamma$ .

**Claim IV:**  $\sigma(e_1, e_i) = \sqrt{2}g(e_i, V)$  for  $i = 2, 3, \dots, n$  on all of  $\gamma$

Using the previous claims

$$\sigma(e_1, e_i) = g(e_1, e_i) + 2g(e_1, V)g(e_i, V) = 0 + \frac{2}{\sqrt{2}}g(e_i, V) = \sqrt{2}g(e_i, V).$$

**Claim V:**  $\sigma(e_i, e_j) = \delta_{ij} + 2g(e_i, V)g(e_j, V)$  for  $i, j = 2, 3, \dots, n$  on all of  $\gamma$

Using the previous claims

$$\sigma(e_i, e_j) = g(e_i, e_j) + 2g(e_i, V)g(e_j, V) = \delta_{ij} + 2g(e_i, V)g(e_j, V).$$

**Claim VI:**  $\det((\sigma(e_i, e_j))_{i,j}) = 1$

Let  $a_i = \sigma(e_i, V)$  for  $i = 1, 2, \dots, n$ . The matrix  $A$  with entries  $\sigma(e_i, e_j)$  can then by the previous claims be written as

$$A = \begin{pmatrix} 1 & \sqrt{2}a_2 & \sqrt{2}a_3 & \cdots \\ \sqrt{2}a_2 & 1 + 2a_2^2 & 2a_2a_3 & \cdots \\ \sqrt{2}a_3 & 2a_2a_3 & 1 + 2a_3^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -\sqrt{2}a_2 & 1 & 0 & 0 & \cdots \\ -\sqrt{2}a_3 & 0 & 1 & 0 & \cdots \\ -\sqrt{2}a_4 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then  $\det B = 1$ . Moreover

$$BA = \begin{pmatrix} 1 & \sqrt{2}a_2 & \sqrt{2}a_3 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hence  $\det(BA) = 1$ . This means that

$$\det A = \frac{\det(BA)}{\det B} = 1,$$

proving the claim.

**Proof part VII(Computation of  $\mathcal{L}_T\omega$ ):** The volume form  $\omega$  induced on  $\mathcal{H}$  by the Riemannian metric  $\sigma$  can be expressed in the frame  $e_1, e_2, \dots, e_n$  as

$$\omega = \sqrt{\det((\sigma(e_i, e_j))_{i,j})} e^1 \wedge e^2 \wedge \cdots \wedge e^n$$

where the  $e^i$  are the covectors defined by  $e^i(e_i) = 1$  and  $e^i(e_j) = 0$  for  $i \neq j$ . By the previous claim the determinant is equal to 1, so

$$\omega = e^1 \wedge e^2 \wedge \cdots \wedge e^n$$

on all of  $\gamma$ . We will use Cartan's formula to compute the Lie derivative  $\mathcal{L}_T\omega$ , so we need to extend the frame  $e_i$  to a neighborhood of  $\gamma$ . Extend  $e_1, \dots, e_n$  to a frame such that  $e_1 = T$ . Extend the dual frame  $e^1, \dots, e^n$  in the natural way by letting

$e^i(e_i) = 1$  and  $e^i(e_j) = 0$  for  $i \neq j$ . Rescale  $e_n$  if necessary so that  $\omega = e^1 \wedge e^2 \wedge \cdots \wedge e^n$  holds everywhere. We will now use this frame to compute  $\mathcal{L}_T \omega$  at the point  $p$ . By Cartan's formula

$$\mathcal{L}_T \omega = i_T d\omega + d(i_T \omega).$$

Since  $\omega$  is an  $n$ -form on an  $n$ -manifold,  $d\omega = 0$ . Hence

$$\mathcal{L}_T \omega = d(i_T \omega).$$

Since  $\omega = e^1 \wedge e^2 \wedge \cdots \wedge e^n$

$$d(i_T \omega) = d(e^1(T)e^2 \wedge \cdots \wedge e^n) = d(e^1(e_1)e^2 \wedge \cdots \wedge e^n) = d(e^2 \wedge \cdots \wedge e^n).$$

Hence

$$\mathcal{L}_T \omega = \sum_{k=2}^n (-1)^k e^2 \wedge \cdots \wedge e^{k-1} \wedge de^k \wedge e^{k+1} \wedge \cdots \wedge e^n.$$

We now compute  $de^k$ , or rather the part of  $de^k$  which does not contain any  $e^j$  for  $j \notin \{1, k\}$ ; all such terms would be annihilated when we insert this expression into the large wedge product above. Since  $de^k$  is a two-form this means that only one of its terms,  $(de^k)(e_1, e_k)e^1 \wedge e^k$ , is interesting. Now

$$(de^k)(e_1, e_k) = e_1(e^k(e_k)) - e_k(e^k(e_1)) - e^k([e_1, e_k]) = -e^k([e_1, e_k])$$

since  $e^k(e_k) = 1$  and  $e^k(e_1) = 0$  everywhere close to  $\gamma$ . We express the Lie bracket, evaluated at the point  $p$ , using the spacetime metric  $g$ :

$$(de^k)(e_1, e_k) = -e^k([e_1, e_k]) = -e^k(\nabla_{e_1} e_k - \nabla_{e_k} e_1) = e^k(\nabla_{e_k} e_1).$$

Recall that  $\nabla$  denotes covariant derivative with respect to  $g$ . We have used that the choice of  $e_2, \dots, e_n$  is such that  $\nabla_{e_1} e_k = 0$  for all  $k$ . We now know that

$$de^k = (e^k(\nabla_{e_k} e_1))e^1 \wedge e^k + \dots$$

where the dots signify terms containing some  $e^j$  for  $j \notin \{1, k\}$ . Hence it holds at the point  $p$  that

$$\begin{aligned} & (-1)^k e^2 \wedge \cdots \wedge e^{k-1} \wedge de^k \wedge e^{k+1} \wedge \cdots \wedge e^n \\ &= (-1)^k e^2 \wedge \cdots \wedge e^{k-1} \wedge (e^k(\nabla_{e_k} e_1))e^1 \wedge e^k \wedge e^{k+1} \wedge \cdots \wedge e^n \\ &= (-1)^k (-1)^{k-2} (e^k(\nabla_{e_k} e_1))e^1 \wedge e^2 \wedge \cdots \wedge e^{k-1} \wedge e^k \wedge e^{k+1} \wedge \cdots \wedge e^n \\ &= (e^k(\nabla_{e_k} e_1))e^1 \wedge e^2 \wedge \cdots \wedge e^n \end{aligned}$$

where the additional factor of  $(-1)^{k-2}$  is due to commuting  $e^1$  with  $e^2, \dots, e^{k-1}$ . Hence

$$\mathcal{L}_T \omega = \sum_{k=2}^n (e^k(\nabla_{e_k} e_1))e^1 \wedge e^2 \wedge \cdots \wedge e^n = \sum_{k=2}^n (e^k(\nabla_{e_k} e_1))\omega.$$

Since  $e_1 = T$  and  $e_2, \dots, e_n$  are  $g$ -orthonormal and  $g$ -orthogonal to  $e_1$ ,

$$\sum_{k=2}^n e^k (\nabla_{e_k} e_1) = \sum_{k=2}^n g(e_k, \nabla_{e_k} e_1) = - \sum_{k=2}^n g(e_k, \nabla_{e_k} (-T)).$$

Recall from Section 1.3.1.1 that the quotient  $T_p \mathcal{H} / \mathbb{R}T$  has an inner product induced by  $g$  such that the image of  $(e_i)_{i=2}^n$  under the projection  $T_p \mathcal{H} \rightarrow T_p \mathcal{H} / \mathbb{R}T$  forms an orthonormal basis. This means that  $\sum_{k=2}^n g(e_k, \nabla_{e_k} (-T))$  is the trace of the null Weingarten map  $b_{-T}$  defined in Section 1.3.1.1. This trace is the null mean curvature  $\theta$  of  $\mathcal{H}$  with respect to  $-T$ . We can then conclude that

$$\mathcal{L}_T \omega = -\theta \omega$$

at  $p$ . Since  $p$  was arbitrary, this completes the proof.  $\square$

The following lemma essentially consists of integrating the Lie derivative  $\mathcal{L}_T \omega$  to relate the null mean curvature  $\theta$  to the Jacobian determinant of the generator flow.

**Lemma 1.3.4.** *Let  $\mathcal{H}$  be a  $C^2$  null hypersurface in a spacetime  $(M, g)$ . Let  $\sigma$  be a Riemannian metric on  $M$  of the form*

$$\sigma(X, Y) = g(X, Y) + 2g(X, V)g(Y, V)$$

*for some  $g$ -unit timelike vector field  $V$  on  $M$ . Let  $T$  be the unique past-directed  $\sigma$ -unit null vector field on  $\mathcal{H}$ , and let  $\theta$  be the null mean curvature of  $\mathcal{H}$  with respect to  $-T$ . Fix  $t > 0$  and let  $\beta_t: \mathcal{H} \rightarrow \mathcal{H}$  denote the flow along  $T$  for time  $t$  (whenever defined). Suppose that  $p$  is such that  $\beta_s(p)$  is defined for all  $s \in [0, t]$ . Let  $J(\beta_t)$  denote the Jacobian determinant of  $\beta_t$  with respect to  $\sigma$ . Then*

$$J(\beta_t)(p) = \exp \left( - \int_0^t \theta(\beta_s(p)) ds \right).$$

*Proof.* Let  $\omega$  denote the volume form of  $\sigma$ . The Jacobian determinant  $J(\beta_t)(p)$  is characterized by

$$\beta_t^*(\omega_{\beta_t(p)}) = J(\beta_t)(p) \omega_p.$$

Let  $\alpha: [0, t] \rightarrow \mathbb{R}$  denote the function such that

$$\beta_s^*(\omega_{\beta_s(p)}) = \alpha(s) \omega_p.$$

Note that  $\alpha(0) = 1$  since  $\beta_0$  is the identity, and that  $\alpha(t) = J(\beta_t)(p)$ . For simplicity of notation, we will abbreviate  $\omega_{\beta_s(p)}$  by  $\omega_s$ . By definition of the Lie derivative

$$(\mathcal{L}_T \omega)_{\beta_s(p)} = \left. \frac{d}{d\tau} \right|_{\tau=0} \beta_{s+\tau}^* \omega_{s+\tau}.$$

Using the function  $\alpha$

$$\beta_\tau^* \omega_{s+\tau} = \beta_{-s}^* \beta_{s+\tau}^* \omega_{s+\tau} = \beta_{-s}^* \alpha(s+\tau) \omega_0 = \alpha(s+\tau) \beta_{-s}^* \omega_0.$$

Note that the pullbacks used in the above computation make sense for sufficiently small  $\tau$  since  $\beta_{-s}^*$  is defined at  $p$  when  $s \in [0, t]$  and  $\beta_{s+\tau}^*$  is defined at  $\beta_{s+\tau}(p)$  when  $s + \tau$  belongs to some sufficiently small open neighborhood of  $[0, t]$ . By definition of  $\alpha$  it holds that  $\beta_s^* \omega_s = \alpha(s) \omega_0$  so that  $\omega_s = \alpha(s) \beta_{-s}^* \omega_0$ . Hence

$$\beta_\tau^* \omega_{\beta_{s+\tau}(p)} = \frac{\alpha(s+\tau)}{\alpha(s)} \omega_s.$$

Returning to the Lie derivative, we now know that

$$(\mathcal{L}_T \omega)_{\beta_s(p)} = \left. \frac{d}{d\tau} \right|_{\tau=0} \frac{\alpha(s+\tau)}{\alpha(s)} \omega_s = \frac{\alpha'(s)}{\alpha(s)} \omega_s.$$

On the other hand, Lemma 1.3.3 tells us that

$$(\mathcal{L}_T \omega)_{\beta_s(p)} = -\theta(\beta_s(p)) \omega_s.$$

Hence

$$\frac{\alpha'(s)}{\alpha(s)} = -\theta(\beta_s(p)).$$

Solving this differential equation subject to the initial condition that  $\alpha(0) = 1$  we see that

$$\alpha(t) = \exp \left( - \int_0^t \theta(\beta_s(p)) ds \right).$$

Since  $\alpha(t) = J(\beta_t)(p)$  this completes the proof.  $\square$

### 1.3.2 Structure of horizons

We begin by defining the abstract concept of a "horizon" (following [8]), and state some previously known results about the regularity of horizons. We then prove that the Cauchy horizons we will work with are horizons in this sense.

#### 1.3.2.1 Abstract horizons

**Definition 1.3.5.** Consider a spacetime. We say that an embedded topological hypersurface is *past null geodesically ruled* if every point on the hypersurface belongs to a past inextendible null geodesic contained in the hypersurface. These geodesics are called *generators*.

**Remark 1.3.6.** Note that if a past null geodesically ruled hypersurface is a  $C^2$  null hypersurface, then these generators are the same as those defined in Section 1.3.1.1.



**Definition 1.3.7.** Consider a spacetime. A *horizon* is an embedded, achronal, past null geodesically ruled, closed (as a set) topological hypersurface.

**Remark 1.3.8.** One may just as well define a horizon to be future null geodesically ruled. Indeed, in [8] the distinction is made between a "past horizon" and a "future horizon". However, since we will work only with future Cauchy horizons it is convenient to restrict our attention to past null geodesically ruled horizons.

**Remark 1.3.9.** If an open subset of a horizon is past null geodesically ruled, then its generators are the restrictions of the generators of the horizon.

Note that we have assumed no smoothness in the definition. Note also that the generators through a point of a horizon are by no means necessarily unique. In fact, we have the following theorem (see Theorem 3.5 in [3] and Proposition 3.4 in [10]).

**Theorem 1.3.10.** *A horizon is differentiable precisely at those points which belong to a single generator.*

We also note that horizons are null hypersurfaces whenever they are differentiable, so that the generators of a  $C^2$  horizon are precisely the integral curves of the null vector fields:

**Proposition 1.3.11.** *If a horizon  $\mathcal{H}$  is differentiable at a point  $p$ , then  $T_p\mathcal{H}$  is a null hyperplane.*

*Proof.* Since  $p$  belongs to a lightlike geodesic segment contained in  $\mathcal{H}$ , we know that  $T_p\mathcal{H}$  contains null vectors. If  $T_p\mathcal{H}$  were to contain a timelike vector, then there would be a timelike curve in  $\mathcal{H}$  with this tangent vector. This would contradict achronality of  $\mathcal{H}$ , and hence  $T_p\mathcal{H}$  must be a null hyperplane.  $\square$

Finally, we note that generators can only intersect in common endpoints.

**Proposition 1.3.12.** *Let  $\mathcal{H}$  be a horizon, and suppose that  $p$  is an interior point of a generator  $\Gamma$ . Then there is no other generator containing  $p$ .*

*Proof.* Suppose that some other generator  $\Gamma'$  contained  $p$ . Let  $q$  be a point to the past of  $p$  along  $\Gamma'$ , and let  $r$  be a point to the future of  $p$  along  $\Gamma$ . By following  $\Gamma'$  from  $q$  to  $p$  and then  $\Gamma$  from  $p$  to  $r$  we have connected  $q$  and  $r$  by a causal curve which is not a null geodesic. By [7, Proposition 2.6.9] this curve cannot be achronal. Since the image of the curve belongs to  $\mathcal{H}$ , this contradicts achronality of  $\mathcal{H}$ .  $\square$

### 1.3.2.2 Cauchy horizons

We now connect the statements in Section 1.3.2.1 about abstract horizons to the particular case of a Cauchy horizon in a spacetime. We begin by quoting [15, Proposition 2.7]. A similar statement can be found in [7, Proposition 2.10.6].

**Proposition 1.3.13.** *Let  $S$  be an achronal subset of a spacetime  $M$ . Then the set  $H^+(S) \setminus \text{edge}(S)$ , if nonempty, is an achronal  $C^0$  hypersurface of  $M$  ruled by null geodesics, each of which either is past inextendible in  $M$  or has a past endpoint on  $\text{edge}(S)$ .*

**Corollary 1.3.14.** *Let  $M$  be a spacetime. Suppose that  $S \subseteq M$  is an achronal set with  $\text{edge}(S) = \emptyset$ . Then  $H^+(S)$  is a horizon in the sense of Definition 1.3.7.*

*Proof.* The proposition tells us that  $H^+(S)$  is a topological hypersurface which is achronal and past null geodesically ruled. To see that  $H^+(S)$  is closed, note that it by definition is the difference of a closed set and an open set. This completes the proof.  $\square$

We conclude with a lemma allowing us to apply Corollary 1.3.14 to closed spacelike hypersurfaces.

**Lemma 1.3.15.** *Let  $M$  be a spacetime and let  $S$  be a spacelike hypersurface which is closed as a set. Then  $\text{edge}(S) = \emptyset$ .*

*Proof.* Let  $p \in S$ . After choosing a sufficiently small neighborhood  $U$  of  $p$ , the set  $S \cap U$  is a Cauchy surface for  $U$ . In particular there is a time function on  $U$  the zero set of which is  $S \cap U$ . By continuity, the time function evaluated along any timelike curve from  $I^-(p, U)$  to  $I^+(p, U)$  must take the value zero, and so every such curve must intersect  $S$ . This means that  $\text{edge}(S) \cap S = \emptyset$ . Since  $S$  is closed and  $\text{edge}(S) \subseteq \overline{S}$  by definition, this completes the proof.  $\square$

### 1.3.3 Tipler's original proof

This section is given only for motivation. No other section depends logically on it.

The following is the theorem about cobordisms which is proved (but not explicitly stated) by Tipler in [32, Theorems 3 and 4]. Tipler did not mention the need for the condition that  $H^+(S_1)$  is  $C^2$ . In the following sections, ending with Theorem 1.3.46, we will prove as a corollary to the smoothness theorem of Section 1.3.5 that this condition is not necessary. However, we begin by correctly stating what Tipler's original proof shows, and summarizing the idea of that proof. We have added some details not present in the original proof, and left others out. For instance, Tipler discusses other energy conditions than the strict lightlike convergence condition, as will we in later sections.

**Theorem 1.3.16.** *Let  $n \geq 2$ , let  $S_1, S_2$  be compact  $n$ -dimensional manifolds and let  $(M, g)$  be a compact connected Lorentzian cobordism between  $S_1$  and  $S_2$  which satisfies the strict lightlike convergence condition. **Suppose moreover that  $H^+(S_1)$  is  $C^2$ .** Then there exists a diffeomorphism  $\varphi: S_1 \times [0, 1] \rightarrow M$  such that the submanifold  $\varphi(\{x\} \times [0, 1])$  is  $g$ -timelike for every  $x \in S_1$ ; in particular,  $S_1$  is diffeomorphic to  $S_2$ .*

*Proof.* For a rigorous proof, it is necessary to embed  $M$  in a manifold without boundary (as we will do in Section 1.3.6.1) to apply propositions about manifolds without boundary. For the purposes of this proof sketch, we ignore this complication.

If  $M$  satisfies the chronology condition then Theorem 1.2.1 gives the desired conclusion. Suppose for contradiction that  $M$  does not satisfy the chronology condition. Then it contains a closed timelike curve  $\gamma$ . We first wish to show that  $H^+(S_1)$  is nonempty. Let  $q \in \text{im } \gamma$ . Note that  $\gamma$  is a timelike past inextendible curve through  $q$  which does not intersect  $S_1$ , so (according to Proposition A.2.13) it is the case that  $q \notin \overline{D^+(S_1)}$ . Let  $\Gamma$  be an arbitrary curve contained in  $M$  from some point of  $S_1 \subseteq D^+(S_1)$  to  $q$ . Suppose that  $\Gamma$  does not contain a point of  $H^+(S_1)$ . By definition  $H^+(S_1) = \overline{D^+(S_1)} \setminus I^-(D^+(S_1))$ , so for each  $r \in \Gamma$  either  $r \notin \overline{D^+(S_1)}$  or  $r \in I^-(D^+(S_1))$ . Let

$$t = \sup\{s \mid \Gamma(t) \in \overline{D^+(S_1)}\}$$

and

$$r = \Gamma(t).$$

Then  $r \in \overline{D^+(S_1)}$  since this set is closed. Suppose to get a contradiction that  $r$  belongs to  $I^-(D^+(S_1))$ . This set is open so there is a point  $r_\epsilon = \Gamma(t + \epsilon)$  for some  $\epsilon > 0$  such that  $r_\epsilon \in I^-(D^+(S_1))$ . Hence there is a point  $p \in D^+(S_1)$  such that there is a timelike curve  $\beta$  from  $r$  to  $p$ . By maximality of  $t$  we know that  $r \notin \overline{D^+(S_1)}$  so there is some past inextendible timelike curve with future endpoint  $r$  which does not meet  $S_1$ . Concatenating  $\alpha$  and  $\beta$  we get a past inextendible timelike curve with future endpoint  $p$  which does not meet  $S_1$ , contradicting that  $p \in D^+(S_1)$ . Hence  $r \notin I^-(D^+(S_1))$ , so by definition of  $H^+(S_1) = \overline{D^+(S_1)} \setminus I^-(D^+(S_1))$  it holds that  $r \in H^+(S_1)$ . Hence  $H^+(S_1) \cap M$  is nonempty.

Since  $H^+(S_1)$  is assumed to be  $C^2$ , it has a  $C^1$  null vector field tangent to the null generators. The null generators are totally past imprisoned in the compact set  $M$ , so by Lemma B.1.1 they are past complete. Let  $\theta$  denote the null mean curvature of  $H^+(S_1)$ . From inequality (1.3) and the lightlike convergence condition we see that if  $\theta > 0$  at some point then  $\theta$  will diverge within a finite affine distance to the past. This would contradict smoothness of  $\theta$ , so we conclude that  $\theta \leq 0$  everywhere.

Let  $V$  be a timelike vector field and introduce the Riemannian metric  $\sigma$  defined by

$$\sigma(X, Y) = g(X, Y) + 2g(X, V)g(Y, V).$$

Let  $\omega$  be the volume form induced on  $H^+(S_1)$  by  $\sigma$  and let  $T$  be the  $\sigma$ -unit past-directed null vector field on  $H^+(S_1)$ . Let  $\beta_t: H^+(S_1) \rightarrow H^+(S_1)$  denote the flow along  $T$ . Then Lemma 1.3.3 tells us that

$$\frac{d}{dt} \int_{\beta_t(H^+(S_1))} \omega = - \int_{H^+(S_1)} \theta \omega.$$

Since  $\theta \leq 0$  we then know that

$$\frac{d}{dt} \int_{\beta_t(H^+(S_1))} \omega \geq 0.$$

However,  $\beta_t$  is defined on all of  $H^+(S_1)$  since the generators of  $H^+(S_1)$  have no past endpoints, and hence  $\beta_t(H^+(S_1)) \subseteq H^+(S_1)$  so that

$$\frac{d}{dt} \int_{\beta_t(H^+(S_1))} \omega \leq 0.$$

Hence it must hold that  $\theta = 0$  on all of  $H^+(S_1)$ . This, together with the strict light-like convergence condition, contradicts inequality (1.3). Hence  $M$  must satisfy the chronology condition, and we are done.  $\square$

### 1.3.4 Properties of nonsmooth horizons

In general, Cauchy horizons are not  $C^2$  hypersurfaces: Figure 1.5 shows an example of a non- $C^2$  Cauchy horizon. This particular example is "almost  $C^2$ " in the sense that it has a dense open subset which is  $C^2$ , so we would expect that many results about  $C^2$  hypersurfaces are applicable to this example. However, it was shown in [10] that Cauchy horizons are not necessarily almost  $C^2$ . This means that the proof of Tipler's theorem needs to be modified to deal with the possibility of horizons of lower regularity. For this reason, and to prove Theorem 1.3.41 about conditions which make Cauchy horizons smooth, we need some results about general horizons. The definitions and results in this section can be found in [8].

**Definition 1.3.17.** Let  $M$  be a spacetime, and let  $(a, b) \times \Sigma \cong \mathcal{O} \subseteq M$  be an open subset such that each slice  $\{t\} \times \Sigma$  is spacelike and each curve  $(a, b) \times \{p\}$  is timelike. Let  $N \subseteq M$  be a hypersurface. A function  $f: \Sigma \rightarrow (a, b)$  is said to be a *graphing function* of  $N$  if  $N \cap \mathcal{O} = \{(f(x), x) \mid x \in \Sigma\}$ .

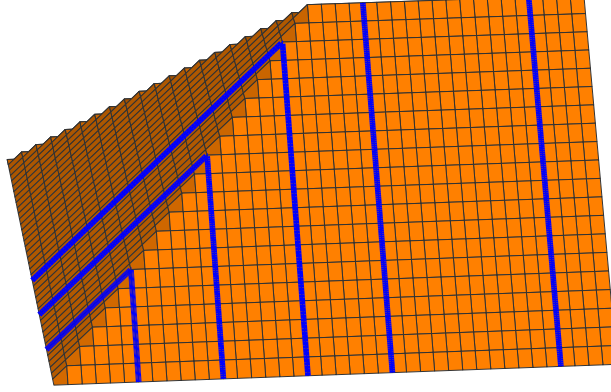


Figure 1.5: Part of the future Cauchy horizon of a spacelike rectangle in  $(2+1)$ -dimensional Minkowski space. Generators of the Cauchy horizon are shown in blue. A more complicated example in which no open subset of the horizon is differentiable is given in [10].

Theorem 2.2 of [8] says that any locally achronal hypersurface, in particular any horizon, is semi-convex. (See Definition A.4.2.) This implies (see [8, Proposition 2.1]) every point on the horizon has a globally hyperbolic spacetime neighborhood  $(-a, a) \times \Sigma$  in which the horizon has a graphing function  $f$  for which there is a subset  $\Sigma_{\mathcal{A}I} \subseteq \Sigma$  such that

1.  $\Sigma \setminus \Sigma_{\mathcal{A}I}$  has measure zero,
2.  $f$  is differentiable at all points of  $\Sigma_{\mathcal{A}I}$ ,
3.  $f$  is *twice-Alexandrov-differentiable* at all points  $x \in \Sigma_{\mathcal{A}I}$ . In other words, there is a quadratic form  $D^2 f(x)$  such that for all  $y \in \Sigma$

$$f(y) - f(x) - df(x)(y - x) = \frac{1}{2} D^2 f(x)(x - y, x - y) + o(|x - y|^2).$$

Moreover, it is shown in [8] that this notion is coordinate invariant: If  $p = (f(x), x)$  with  $x \in \Sigma_{\mathcal{A}I}$  for one globally hyperbolic neighborhood of  $p$ , then  $p = (\tilde{f}(\tilde{x}), \tilde{x})$  with  $\tilde{x} \in \tilde{\Sigma}_{\mathcal{A}I}$  for any other globally hyperbolic neighborhood  $(-\tilde{a}, \tilde{a}) \times \tilde{\Sigma}$  of  $p$ , with corresponding graphing function  $\tilde{f}$ . Hence the following definition makes sense.

**Definition 1.3.18.** Let  $\mathcal{H}$  be a horizon in a spacetime. Denote by  $\mathcal{H}_{\mathcal{A}I}$  the set of all points  $p \in \mathcal{H}$  which are images under a graphing function of one of the corresponding sets  $\Sigma_{\mathcal{A}I}$ . We will call  $\mathcal{H}_{\mathcal{A}I}$  the set of *Alexandrov points* of the horizon.

**Remark 1.3.19.** By the definition of semi-convexity a semi-convex function is the sum of a  $C^2$  function and a convex function, and hence locally Lipschitz. This means that horizons are Lipschitz hypersurfaces.

We will now define the null mean curvature  $\theta_{\mathcal{H}}$  and the null second fundamental form  $B_{\mathcal{H}}$  of  $\mathcal{H}$ , following [8]. More precisely, we will define  $\theta_{\mathcal{H}}$  and  $B_{\mathcal{H}}$  on the intersection of  $\mathcal{H}$  with a globally hyperbolic coordinate neighborhood  $\mathcal{O}$ . This definition is *not* coordinate invariant. However,  $\theta_{\mathcal{H}}$  and  $B_{\mathcal{H}}$  are defined up to pointwise scaling by a positive function, so the sign of  $\theta_{\mathcal{H}}$  is globally well defined.

**Definition 1.3.20.** Let  $\mathcal{H}$  be a horizon in a spacetime  $(M, g)$ . Choose a globally hyperbolic coordinate neighborhood  $\mathcal{O} = (-a, a) \times \Sigma$  of some point in  $\mathcal{H}$ , and let  $f: \Sigma \rightarrow (-a, a)$  be the graphing function of  $\mathcal{H}$  in this neighborhood. Let the function  $t: (-a, a) \times \Sigma \rightarrow (-a, a)$  be the projection. For each point  $p \in \mathcal{O} \cap \mathcal{H}$ , with  $x$  such that  $p = (f(x), x)$ , define  $k(p) = -dt + df(x)$ . This makes sense since  $f$  is differentiable at all such points. Let  $K$  be the vector field dual to  $k$  with respect to  $g$ . Let  $e_0 \in T_p \mathcal{O}$  be the vector which is  $g$ -dual to  $dt$ . Choose a basis  $e_1, \dots, e_n$  for  $T_p \mathcal{H}$  such that

- $e_n = K_p$ ,
- $g(e_i, e_i) = 1$  if  $1 \leq i \leq n-1$ ,
- $g(e_i, e_j) = 0$  if  $i \neq j$ ,
- $g(e_i, e_0) = 0$  if  $1 \leq i \leq n-1$ .

We now define  $\theta_{\mathcal{H}}$  and  $B_{\mathcal{H}}$  using the coordinate formulae

$$\theta_{\mathcal{H}} = \sum_{l=1}^{n-1} e_l^i e_l^j \left( D_{ij}^2 f - \Gamma_{ij}^\mu k_\mu \right),$$

$$B_{\mathcal{H}}(X^a e_a, Y^b e_b) = X^a Y^b e_a^i e_b^j \left( D_{ij}^2 f - \Gamma_{ij}^\mu k_\mu \right).$$

In the definitions above, we have constructed an "artificial" covariant derivative  $\nabla_i k_j \approx D_{ij}^2 f - \Gamma_{ij}^\mu k_\mu$  using the Alexandrov second derivative  $D^2 f$  of  $f$ . We emphasize again that this definition of  $\theta_{\mathcal{H}}$  is not independent of the coordinate system. However, the definitions using different coordinate systems differ only by a positive multiplicative constant. In particular, the sign of  $\theta_{\mathcal{H}}$  is invariantly defined. (See [8, Proposition 2.5].)

**Remark 1.3.21.** If  $\mathcal{H}$  is  $C^2$ , then  $\theta_{\mathcal{H}}$  and  $B_{\mathcal{H}}$  agree with the null mean curvature  $\theta_K$  and null second fundamental form  $B_K$  defined in Section 1.3.1.1.

**Remark 1.3.22.** By [8, Theorem 5.1] the  $(1, 1)$  tensor  $b_{AI}$  associated to  $B_{AI}$  satisfies equation (1.1) from Section 1.3.1.1.

We will later derive formulae involving  $\theta_{AI}$  for the area of a horizon, and knowing the sign of  $\theta_{AI}$  will yield inequalities between different areas. In our case, the generators of the horizon will be past complete, so we will have use of the following result. It is a generalization of [8, Proposition 4.17], and the proof of that proposition is sufficient for proving the generalization as well, since the proof is local.

**Proposition 1.3.23.** *Let  $M$  be a spacetime, and let  $\mathcal{H}$  be a horizon in  $M$ . Let  $\hat{\mathcal{H}}$  be an open subset of  $\mathcal{H}$  which is also past null geodesically ruled. Suppose that the generators of  $\hat{\mathcal{H}}$  are complete in the past direction and that the null energy condition holds. Then*

$$\theta_{AI} \leq 0 \quad \text{on } \mathcal{H}_{AI} \cap \hat{\mathcal{H}}.$$

**Remark 1.3.24.** The result in [8] is expressed with the opposite time orientation compared to our setting. Consequently we obtain the inequality  $\theta_{AI} \leq 0$  instead of  $\theta_{AI} \geq 0$ .

### 1.3.5 A smoothness theorem

In [5, Section 4] the question was posed whether a compact Cauchy horizon is necessarily smooth. A negative answer was given by the same authors in [6, Section 4], where it was also mentioned that compactness together with some energy condition might be sufficient to guarantee smoothness. In this section, we show using methods from [8] that this is indeed the case.

#### 1.3.5.1 Outline of the proof

The theorem which will be proved in this section is Theorem 1.3.41, stating that compact horizons in a spacetime which satisfies the null energy condition are smooth. We first give an outline of the proof. Horizons are Lipschitz null hypersurfaces, and so differentiable almost everywhere. At the points of differentiability there is a unique (up to scaling) null tangent vector, giving rise to an almost everywhere defined vector field on the horizon. By restricting to a suitably chosen subset of the horizon, we may define a flow along this vector field. One may then construct a  $C^{1,1}$  manifold containing, locally, this chosen subset, and extend the flow to a Lipschitz flow on the  $C^{1,1}$  manifold. This is sufficient regularity to express how the area of a set changes under the flow, and this change of area is the central idea of the proof. To measure area, we introduce a Riemannian metric  $\sigma$  on the spacetime. With a suitably chosen such metric, the change in area is related to the

Alexandrov null mean curvature  $\theta_{\mathcal{A}I}$  of the Cauchy horizon. The argument for this relation between area change and  $\theta_{\mathcal{A}I}$  proceeds via a  $C^2$  approximation of a part of the local  $C^{1,1}$  approximation of the original horizon. Once the relation between  $\theta_{\mathcal{A}I}$  and area change has been established, knowledge of the sign of  $\theta_{\mathcal{A}I}$  gives an inequality for area change under the flow. A sufficient condition under which the sign of  $\theta_{\mathcal{A}I}$  may be determined is that all null geodesics in the horizon are complete in the past direction, together with an energy condition. That the generators are complete follows from Lemma B.1.1. By these arguments, we determine that the flow increases area. However, the flow maps a subset of the horizon into itself, thereby decreasing area. Hence the only possibility is that the flow conserves area. We show in Lemma 1.3.39 that this condition implies that the horizon is smooth.

### 1.3.5.2 Flow sets and generator flow

We wish to generalize the notion of the generator flow on  $C^2$  null hypersurfaces discussed in Section 1.3.1.2 to possibly nonsmooth horizons. In other words, we want a flow along generators of a horizon  $\mathcal{H}$ . However, since some points belong to several generators it is in general not possible to do this on all of  $\mathcal{H}$ . Instead, we construct a smaller subset on which to define the flow.

**Definition 1.3.25.** Let  $\mathcal{H}$  be a horizon in a spacetime  $M$ . Define the *total flow set* of  $\mathcal{H}$  to be the set  $A_0(\mathcal{H})$  of points  $p \in \mathcal{H}$  such that the following conditions are satisfied:

- There is a unique generator  $\Gamma$  of  $\mathcal{H}$  passing through  $p$ .
- The point  $p$  belongs to the interior of  $\Gamma$ .
- Each interior point of  $\Gamma$  is an Alexandrov point.

Let  $\sigma$  be a Riemannian metric on  $M$ . For  $\delta > 0$  define the  $\delta$ -flow set of  $\mathcal{H}$  with respect to  $\sigma$  to be the set

$$A_\delta(\mathcal{H}, \sigma) = \{p \in A_0(\mathcal{H}) \mid \text{The generator through } p \text{ exists} \\ \text{for a } \sigma\text{-distance greater than } \delta \\ \text{to the past and to the future}\}.$$

**Remark 1.3.26.** Note that the total flow set is the union of all  $\delta$ -flow sets:

$$A_0(\mathcal{H}) = \bigcup_{\delta > 0} A_\delta(\mathcal{H}, \sigma).$$

**Remark 1.3.27.** When the context allows it, we will sometimes drop  $\mathcal{H}$  and  $\sigma$  from the notation and write simply  $A_0$  or  $A_\delta$ .



For the next proposition, we will need the following result, a proof of which can be found in [8, Theorem 5.6].

**Lemma 1.3.28.** *Let  $\mathcal{H}$  be a horizon in a spacetime  $M$  of dimension  $n + 1$ . Suppose that  $\mathcal{S}$  is a  $C^2$  hypersurface intersecting  $\mathcal{H}$  properly transversally (in the sense that if  $q \in \mathcal{S} \cap \mathcal{H}$  and the tangent space  $T_q \mathcal{H}$  exists then  $T_q \mathcal{S}$  is transverse to  $T_q \mathcal{H}$ ). Define*

$$S_0 = \{q \in \mathcal{S} \cap \mathcal{H} \mid q \text{ is an interior point of a generator of } \mathcal{H}\},$$

$$S_1 = \{q \in S_0 \mid \text{all interior points of the generator through } q \text{ are Alexandrov points of } \mathcal{H}\}.$$

*Then  $S_1$  has full  $(n - 1)$ -dimensional Hausdorff measure in  $S_0$ .*

**Proposition 1.3.29.** *Let  $\mathcal{H}$  be a horizon in a spacetime  $(M, g)$  of dimension  $n + 1$ , and let  $\sigma$  be a Riemannian metric on  $M$ . Let  $\mathfrak{h}^n$  be the  $n$ -dimensional Hausdorff measure induced by the distance function induced by  $\sigma$ . Then the total flow set  $A_0$  of  $\mathcal{H}$  has full  $\mathfrak{h}^n$ -measure in the sense that*

$$\mathfrak{h}^n(\mathcal{H} \setminus A_0) = 0.$$

*Proof.* To show that  $\mathcal{H} \setminus A_0$  has measure zero, it is sufficient to show that each point  $p \in \mathcal{H}$  has an open neighborhood  $U \subseteq M$  such that  $\mathfrak{h}^n(U \cap (\mathcal{H} \setminus A_0)) = 0$ , for  $\mathcal{H}$  can be covered by countably many such neighborhoods since it is second-countable. The idea of the proof is to construct a spacetime of one dimension greater than  $M$  and apply Lemma 1.3.28 in this higher-dimensional setting.

To this end, choose a globally hyperbolic neighborhood  $U \subseteq M$  of  $p \in \mathcal{H}$  diffeomorphic to  $(-a, a) \times \Sigma$ , where  $\Sigma \subseteq \mathbb{R}^n$ . With this notation, we mean that each slice  $\{t\} \times \Sigma$  is spacelike. We may choose the zero slice to be such that  $p \in \{0\} \times \Sigma$ . Let  $\widehat{M} = M \times I$  denote the product manifold with the metric  $\widehat{g} = g + ds^2$ , where  $s$  refers to the coordinate in the open interval  $I$ . Let  $\widehat{\mathcal{H}} = \mathcal{H} \times I$ . Let  $\pi: \widehat{M} \rightarrow M$  denote the projection. We now verify that  $\widehat{\mathcal{H}}$  is a horizon in  $\widehat{M}$ , which follows easily from  $\mathcal{H}$  being a horizon in  $M$ . Since  $\mathcal{H}$  is an embedded topological hypersurface, so is  $\widehat{\mathcal{H}}$ . By definition of the product topology,  $\widehat{\mathcal{H}} = \pi^{-1}(\mathcal{H})$  is closed. If there were some timelike curve  $\gamma$  between two points of  $\widehat{\mathcal{H}}$ , then  $\pi \circ \gamma$  would be a timelike curve between two points of  $\mathcal{H}$  contradicting achronality of  $\mathcal{H}$ , so  $\widehat{\mathcal{H}}$  must also be achronal. (To see that  $\pi \circ \gamma$  is indeed timelike, note that by definition of  $\widehat{g}$  it holds that  $g(\pi_* V) \leq \widehat{g}(V)$  for all vectors  $V$ .) Finally  $\widehat{\mathcal{H}}$  is past null geodesically ruled since if  $\Gamma$  is a past inextendible null  $M$ -geodesic contained in  $\mathcal{H}$ , then  $\Gamma \times \{s\}$  is a past inextendible null  $\widehat{M}$ -geodesic contained in  $\widehat{\mathcal{H}}$  for each  $s \in I$ .

We now wish to construct, after possibly decreasing  $a$  or shrinking  $I$ , a diffeomorphism  $\rho: I \rightarrow (-a, a)$  such that the hypersurface

$$\mathcal{S} := \{(t, q, s) \in (-a, a) \times \Sigma \times I \mid t = \rho(s)\}$$

is spacelike. A possible choice of basis for the tangent space  $T_{(t,q,s)}\mathcal{S} \subseteq T_q\mathbb{R} \times \Sigma \times \mathbb{R}$  of  $\mathcal{S}$  at some point  $(t, q, s)$  consists of a basis for the tangent space of  $\Sigma$  together with the vector  $(\rho'(s), 0, 1)$ . The basis of  $T_q\Sigma$  consists of spacelike vectors since  $\Sigma$  is spacelike, and if  $\rho'(s)$  is sufficiently close to zero then  $(\rho'(s), 0, 1)$  is also spacelike (since  $(0, 0, 1)$  is spacelike by definition of  $\widehat{g}$ , and the set of spacelike vectors at a point is open). This means that for each  $(t, q) \in (-a, a) \times \Sigma$ , there is some  $c(t, q) > 0$  such that if  $\zeta < c(t, q)$  then for any  $s \in I$  the vector  $(0, \zeta, 1) \in T_{(t,q,s)}$  is spacelike. Since  $g$  is smooth,  $c$  can be chosen smooth. Hence  $c$  takes some minimum on every compact subset of  $(-a, a) \times \Sigma$ . This minimum is positive since  $c$  is positive on  $(-a, a) \times \Sigma$ , after possibly shrinking  $\Sigma$  and  $a$ . We may then find some real number  $\zeta$  such that  $0 < \zeta < c(t, q)$  for all  $(t, q) \in (-a, a) \times \Sigma$ . Letting  $\rho(s) = \zeta(s - s_0)$  where  $s_0$  is the midpoint of  $I$ , and subsequently shrinking  $I$  or  $a$  to make  $\rho$  bijective, we have found a diffeomorphism  $\rho$  making  $\mathcal{S}$  spacelike. Since  $\rho$  is a diffeomorphism, the restriction of the projection  $\pi: \widehat{M} \rightarrow M$  to  $\mathcal{S}$  is also a diffeomorphism.

Now  $\mathcal{S}$  is a smooth hypersurface in  $\widehat{M}$ , which intersects  $\widehat{\mathcal{H}}$  properly transversally in the sense that if  $q \in \mathcal{S} \cap \widehat{\mathcal{H}}$  and the tangent space  $T_q\widehat{\mathcal{H}}$  exists then  $T_q\mathcal{S}$  is transverse to  $T_q\widehat{\mathcal{H}}$ . Let

$$\widehat{S}_0 = \{q \in \mathcal{S} \cap \widehat{\mathcal{H}} \mid q \text{ is an interior point of a generator of } \widehat{\mathcal{H}}\},$$

$$S_0 = \{q \in U \cap \mathcal{H} \mid q \text{ is an interior point of a generator of } \mathcal{H}\}.$$

Note that  $\pi(\widehat{S}_0) = S_0$  since if  $p$  is an interior point of a generator  $\Gamma$  then  $\pi(p)$  is an interior point of the generator  $\pi(\Gamma)$  and vice versa. Note further that it holds that  $\pi(\mathcal{S} \cap \widehat{\mathcal{H}}) = U \cap \mathcal{H}$ . Moreover, the projection  $\pi$  restricted to  $\mathcal{S}$  is bijective and hence  $\pi((\mathcal{S} \cap \widehat{\mathcal{H}}) \setminus \widehat{S}_0) = (U \cap \mathcal{H}) \setminus S_0$ .

Since  $\pi$  restricted to  $\mathcal{S}$  is a diffeomorphism, both  $\pi|_{\mathcal{S}}$  and its inverse  $(\pi|_{\mathcal{S}})^{-1}$  are locally Lipschitz so that  $\mathfrak{h}^n((\mathcal{S} \cap \widehat{\mathcal{H}}) \setminus \widehat{S}_0) = 0$  if and only if  $\mathfrak{h}^n((U \cap \mathcal{H}) \setminus S_0) = 0$ . The latter set  $(U \cap \mathcal{H}) \setminus S_0$  is the set of endpoints of generators of  $\mathcal{H}$  contained in  $U$ . It is shown in [9] that this set has zero  $\mathfrak{h}^n$ -measure. This means that we can conclude that  $\mathfrak{h}^n((\mathcal{S} \cap \widehat{\mathcal{H}}) \setminus \widehat{S}_0) = 0$ . In other words,  $\widehat{S}_0$  has full measure in  $\mathcal{S} \cap \widehat{\mathcal{H}}$ .

Let

$$\begin{aligned} \widehat{S}_1 = \{q \in \widehat{S}_0 \mid & \text{all interior points of the generator} \\ & \text{through } q \text{ are Alexandrov points of } \widehat{\mathcal{H}}\}. \end{aligned}$$

By Lemma 1.3.28  $\widehat{S}_1$  has full  $\mathfrak{h}^n$  measure in  $S_0$ . Hence it also has full  $\mathfrak{h}^n$ -measure in  $\mathcal{S} \cap \widehat{\mathcal{H}}$ . Since  $\pi$  is bi-Lipschitz,  $\pi(\widehat{S}_1)$  has full  $\mathfrak{h}^n$ -measure in  $U \cap \mathcal{H}$ .

The projection  $\pi: \widehat{M} \rightarrow M$  maps generators to generators, and Alexandrov points of  $\widehat{\mathcal{H}}$  to Alexandrov points of  $\mathcal{H}$ , so each point of  $\widehat{S}_1$  belongs to  $A_0$ . We have then shown that  $A_0 \cap U$  contains a subset  $\pi(\widehat{S}_1)$  which has full measure in  $\mathcal{H} \cap U$ . Hence  $A_0$  itself has full measure in  $\mathcal{H} \cap U$ . As noted in the beginning of the proof,

$\mathcal{H}$  may be covered by countably many such sets  $U$ , so we have shown that  $A_0$  has full  $\mathfrak{h}^n$ -measure in  $\mathcal{H}$ . This completes the proof.  $\square$

**Definition 1.3.30.** Let  $\mathcal{H}$  be a horizon in a spacetime  $(M, g)$ , and let  $A_0$  be its total flow set. Let  $\sigma$  be a Riemannian metric on  $M$ . Since  $\mathcal{H}$  is differentiable at all points in  $A_0$ , there is a unique  $\sigma$ -unit past-directed null vector tangent to  $\mathcal{H}$  at each point in  $A_0$ . This defines a vector field  $T$  on  $A_0$ , which is tangent to the generators of  $\mathcal{H}$ . Recall that  $A_0$  contains full generators, and hence full integral curves of  $T$ . We will call the flow of  $T$ , whenever it is defined, the *generator flow* of  $\mathcal{H}$  with respect to  $\sigma$ , and denote it by  $(t, p) \mapsto \beta_t(p)$ .

The choice of  $A_0$  was made so that  $A_0$  flows into itself, in the sense that if  $p \in A_0$  and  $t \geq 0$  are such that  $\beta_t(p)$  is defined, then  $\beta_t(p) \in A_0$ .

### 1.3.5.3 Generator flow is area-preserving

The purpose of this section is to prove that the generator flow on a horizon with respect to a certain family of Riemannian metrics preserves the associated Hausdorff measure if the null mean curvature is non-positive. Our immediate goal is to construct a  $C^{1,1}$  approximation of the horizon to be able to express the volume change. We do this in Lemma 1.3.32 and Lemma 1.3.33. We then construct a  $C^2$  approximation of the horizon to compute the volume change in Proposition 1.3.35 and Proposition 1.3.38. The complicated constructions necessary are contained in Lemma 1.3.37.

We begin by stating an extension theorem, which is proved in [8, Proposition 6.6].

**Lemma 1.3.31.** *Let  $B \subseteq \mathbb{R}^n$  be an arbitrary subset and  $f: B \rightarrow \mathbb{R}$  be an arbitrary function. Suppose that there is some constant  $C > 0$ , and some function  $B \rightarrow \mathbb{R}^n$ ,  $p \mapsto a_p$  (not necessarily continuous) such that the following two conditions hold:*

1.  *$f$  has global upper and lower support paraboloids of opening  $C$ . Explicitly, for all  $x, p \in B$ ,*

$$|f(x) - f(p) - \langle x - p, a_p \rangle| \leq C\|x - p\|^2.$$

2. *The upper and lower support paraboloids of  $f$  are disjoint. Explicitly, for all  $p, q \in B$  and all  $x \in \mathbb{R}^n$ ,*

$$f(p) + \langle x - p, a_p \rangle - C\|x - p\|^2 \leq f(q) + \langle x - q, a_q \rangle - C\|x - q\|^2.$$

*Then there is a function  $F: \mathbb{R}^n \rightarrow \mathbb{R}$  of class  $C_{loc}^{1,1}$  such that  $f$  is the restriction of  $F$  to  $B$ .*

Using this lemma, we may prove the following.

**Lemma 1.3.32.** *Let  $\mathcal{H}$  be a horizon in an  $(n+1)$ -dimensional spacetime  $M$ . Let  $\sigma$  be any Riemannian metric on  $M$ , let  $\delta > 0$  and let  $A_\delta$  be the  $\delta$ -flow set of  $\mathcal{H}$  with respect to  $\sigma$ . Let  $p \in A_\delta$ . Then there is some open globally hyperbolic neighborhood  $V \subseteq M$  of  $p$  and a  $C^{1,1}$  hypersurface  $N \subseteq V$  in  $M$  such that  $A_\delta \cap V \subseteq N$ .*

*Proof.* For each point  $q \in A_\delta$ , let  $q^+$  denote the point a  $\sigma$ -distance  $\delta$  to the future along the unique generator through  $q$ . Similarly, let  $q^-$  denote the point along the generator a distance  $\delta$  to the past. By one of the defining properties of  $A_\delta$ , we have  $q^+, q^- \in \mathcal{H}$ .

By the same reasoning as is used in the proof of Lemma 6.9 in [8], one may obtain a globally hyperbolic neighborhood  $V \subseteq W$  of  $p$  and a constant  $C > 0$  with the following properties:

- $V$  is diffeomorphic to  $(-a, a) \times B^n(r)$  with the slices  $\{t\} \times B^n(r)$  spacelike and the curves  $(-a, a) \times \{x\}$  timelike and future-directed for all  $t \in (-a, a)$  and all  $x \in B^n(r)$ .
- Let  $f$  denote the graphing function of the horizon over  $B^n(r)$ , i.e. the function such that  $V \cap \mathcal{H} = \{(f(x), x) \mid x \in B^n(r)\}$ . For each  $q = (f(x_q), x_q) \in V \cap A_\delta$ , the graph of the function

$$f_q^-(x) = f(x_q) + df(x_q)(x - x_q) - C\|x - x_q\|^2,$$

with the exception of the point  $q = (f(x_q), x_q)$  itself, lies in the timelike past  $I^-(q^+, V)$  of  $q^+$ .

- For each  $q = (f(x_q), x_q) \in V \cap A_\delta$ , the graph of the function

$$f_q^+(x) = f(x_q) + df(x_q)(x - x_q) + C\|x - x_q\|^2,$$

with the exception of the point  $q = (f(x_q), x_q)$  itself, lies in the timelike future  $I^+(q^-, V)$  of  $q^-$ .

Note that if this holds for some value of  $C$ , it holds for all larger values of  $C$  as well.

We will show that these conditions imply the first hypothesis of Lemma 1.3.31. Suppose that the condition is violated. Then either  $f(x) > f_q^+(x)$  or  $f(x) < f_q^-(x)$  for some  $q = (f(x_q), x_q) \in A_\delta$  and  $x \in B^n(r)$ . The argument is the same for both cases, so suppose without loss of generality that the first is the case. Since  $f(x_q) = f_q^+(x_q)$  we must have  $x \neq x_q$ . Then  $(f_q^+(x), x)$  belongs to the timelike future of  $q^-$ , by the choice of  $C$ . However, since  $f(x) > f_q^+(x)$ , the point  $(f(x), x)$  lies to the timelike future of  $(f_q^+(x), x)$ . This means that we can connect  $q^-$  to  $(f_q^+(x), x)$  to

$(f(x), x)$  by a timelike curve. Hence  $(f(x), x)$  belongs to the timelike future of  $q^-$ . Since both points belong to the horizon, this violates achronality of the horizon. This proves the first hypothesis of Lemma 1.3.31.

For the second hypothesis, note that the first continues to hold if we increase  $C$ . By making sure that  $C$  is sufficiently large compared to the Lipschitz constant of  $f$  and the values of  $f$ , one may conclude as in the proof of Lemma 6.9 in [8] that the second hypothesis is satisfied as well.

Let  $B$  denote the projection of  $A_\delta \cap V$  on  $B^n(r)$ . We can then apply the extension theorem described in Lemma 1.3.31 to obtain a  $C^{1,1}$  extension  $\mathbb{R}^n \rightarrow \mathbb{R}$  of  $f|_B : B \rightarrow (-a, a)$ . Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  denote the restriction to  $B^n(r)$  of this extension. By definition  $F$  agrees with  $f$  on  $B$ . In particular, the graph of  $F$  contains  $p = (f(x_p), x_p)$ , so  $F(x_p) \in (-a, a)$ . Since  $F$  is continuous, there is some neighborhood  $B^n(\epsilon) \subseteq B^n(r)$  of  $x_p$  such that  $F(B(\epsilon)) \subseteq (-a, a)$ . Hence by shrinking the neighborhood  $V$  to  $(-a, a) \times B^n(\epsilon)$  and letting  $N$  be the graph of  $F$  there, we have obtained a  $C^{1,1}$  hypersurface containing  $A_\delta \cap V$ .  $\square$

**Lemma 1.3.33.** *Let  $\mathcal{H}$  be a horizon in an  $(n+1)$ -dimensional spacetime  $(M, g)$  equipped with a Riemannian metric  $\sigma$ , let  $\delta > 0$ , let  $A_\delta$  be the  $\delta$ -flow set of  $\mathcal{H}$  with respect to  $\sigma$ , let  $\tilde{A}_\delta$  be the full-density subset (in the sense of Definition A.4.9) of  $A_\delta$ , let  $V$  be a globally hyperbolic open neighborhood of  $p$  and let  $N \subseteq V$  be a  $C^{1,1}$  hypersurface containing  $A_\delta \cap V$ , which can be represented by a graphing function in  $V$ .*

*Fix  $t \geq 0$ . Let  $\beta_t : A_\delta \cap V \rightarrow A_0$  be the restriction of the generator flow (with respect to  $\sigma$ ) to  $A_\delta \cap V$ , and suppose that this flow is defined on all of  $A_\delta \cap V$ . Then there is a neighborhood  $U \subseteq V$  of  $p$  such that the restriction of  $\beta_t$  to  $\tilde{A}_\delta \cap U$  is the restriction of a locally Lipschitz function  $\widehat{\beta}_t : N \cap U \rightarrow M$ .*

*Proof.* Let  $(a, b) \times \Sigma$  be a decomposition in space and time of the globally hyperbolic neighborhood  $V$  of  $p$ , and let  $f$  denote the graphing function of  $N$  with respect to this decomposition.

We wish to construct a Lipschitz normal vector field to  $N$  in a neighborhood of  $p = (f(x), x)$ . Choose a frame  $(e_i)_{i=1}^n$  close to  $p$  consisting of the pushforward of a frame of  $\Sigma$  close to  $x$  under the map  $y \mapsto (f(y), y)$ . This frame is Lipschitz since  $f$  is  $C^{1,1}$ . By shrinking  $V$  we may assume that the frame covers all of  $N$ . The condition that a vector field  $\mathbf{n}$  along  $N$  is normal to  $N$  with respect to the spacetime metric  $g$ , and consists of unit vectors with respect to the Riemannian metric  $\sigma$  can be

expressed by saying that  $\mathbf{n}$  satisfies the  $n + 1$  equations

$$\begin{aligned}\sigma(\mathbf{n}, \mathbf{n}) - 1 &= 0, \\ g(e_1, \mathbf{n}) &= 0 \\ g(e_2, \mathbf{n}) &= 0 \\ &\vdots \\ g(e_n, \mathbf{n}) &= 0.\end{aligned}$$

Trivialize  $TN$  using the frame  $(e_i)_{i=0}^n$ . Define  $F: N \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$  by the above equations. Explicitly

$$F(\mathbf{n}) = (\sigma(\mathbf{n}, \mathbf{n}) - 1, g(e_1, \mathbf{n}), \dots, g(e_n, \mathbf{n})).$$

Let  $\mathbf{n}$  be a zero of  $F$ . The tangent map of  $F$  at  $\mathbf{n}$  with respect to the  $\mathbb{R}^{n+1}$  component is

$$\mathbf{k} \mapsto (2\sigma(\mathbf{n}, \mathbf{k}), g(e_1, \mathbf{k}), \dots, g(e_n, \mathbf{k})).$$

We wish to show that this tangent map has full rank. For dimensional reasons, this is equivalent to its kernel being trivial. If  $\mathbf{k}$  belongs to the kernel, then  $g(e_i, \mathbf{k}) = 0$  for all  $1 \leq i \leq n$ . Hence  $\mathbf{k}$  is a normal vector to  $N$ , and hence parallel to  $\mathbf{n}$ . If  $\mathbf{k}$  belongs to the kernel of the tangent map then it also holds that  $\sigma(\mathbf{n}, \mathbf{k}) = 0$ . When  $\mathbf{k}$  is parallel to  $\mathbf{n}$ , this can only happen when  $\mathbf{k} = 0$ . This shows that the tangent map of the  $\mathbb{R}^{n+1}$  component of  $F$  has full rank. Clearly  $N$  has a tangent vector  $\mathbf{n}$  which is a zero of  $F$  at the point  $p$ . This means that we can apply Clarke's Implicit Function Theorem (Corollary, p. 256 in [11]) to conclude that there is a Lipschitz function  $\mathbf{n}$  satisfying  $F(q, \mathbf{n}(q)) = 0$  in a neighborhood of  $p$ . By shrinking  $V$  if necessary, we have then found a Lipschitz normal (with respect to  $g$ ) vector field to  $N$  which is of unit length (with respect to  $\sigma$ ).

By considering graphing functions of  $N$  and  $\mathcal{H}$  and applying the result about tangent spaces at full-density points described in Proposition A.4.12 we see that the tangent spaces  $T_q N$  and  $T_q \mathcal{H}$  agree at all  $q \in \tilde{A}_\delta$ . Consider now a point  $q \in \tilde{A}_\delta$ . By Proposition 1.3.11 the tangent space  $T_q \mathcal{H}$  is a null hyperplane. Since  $q \in \tilde{A}_\delta$  we have  $T_q N = T_q \mathcal{H}$ , so  $T_q N$  is also a null hyperplane. The normal vector  $\mathbf{n}_q$  to the null hyperplane  $T_q N$  is then null, and any two null vectors in a null hyperplane are parallel, so the vector  $\mathbf{n}_q$  for a point  $q \in \tilde{A}_\delta$  is parallel to any tangent vector of the null geodesic generator through  $q$ .

Once again, consider some point  $q \in \tilde{A}_\delta \subseteq N$ . Since  $\beta_t(q)$  is a point along the geodesic (with respect to the spacetime metric  $g$ ) with initial velocity parallel to  $\mathbf{n}_q$  (which by definition is nonzero) it holds that there is some function  $r: \tilde{A}_\delta \rightarrow \mathbb{R}$  such that

$$\beta_t(q) = \exp^g(r(q)\mathbf{n}_q).$$

For each  $q \in N$  there is a unique real number  $\hat{r}(q)$  such that the  $\sigma$ -distance between  $q$  and  $\exp^g(\hat{r}(q)\mathbf{n}_q)$  along the curve  $\tau \rightarrow \exp^g(\tau\mathbf{n}_q)$  is precisely  $t$ . By definition  $r$  and  $\hat{r}$  coincide on  $\tilde{A}_\delta$ .

We now want to use Clarke's implicit function theorem (Corollary, p. 256 in [11]) to conclude that  $\hat{r}$  is locally Lipschitz. By definition, the choice  $\xi = \hat{r}$  solves the equation

$$\forall q \in N \quad \int_0^1 \left\| \frac{\partial}{\partial \tau} \exp^g(\tau \xi(q)\mathbf{n}_q) \right\|_\sigma d\tau = 1,$$

and this solution is of course unique. In other words  $\xi = \hat{r}$  is the function satisfying  $F(q, \xi(q)) = 0$  where  $F: N \times \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$F(q, t) = \left( \int_0^1 \left\| \frac{\partial}{\partial \tau} \exp^g(\tau t \mathbf{n}_q) \right\|_\sigma d\tau \right) - 1.$$

Since  $\mathbf{n}$  is locally Lipschitz, so is  $F$ . Note that  $F$  has a partial derivative with respect to  $t$  and that  $\frac{\partial F}{\partial t} \neq 0$  everywhere since  $\mathbf{n}$  is nowhere zero. Clarke's implicit function theorem now tells us that there is a solution  $\xi$  of  $F(q, \xi(q)) = 0$  with  $\xi(p) = r(p)$  which is Lipschitz in a neighborhood of  $p$ . Since we already know that the only solution of this equation is  $\hat{r}$ , this shows that  $\hat{r}$  is locally Lipschitz in some neighborhood  $U$  of  $p$ .

Since  $\hat{r}$  is Lipschitz on  $U$ , the function  $\widehat{\beta}_t: N \cap U \rightarrow M$  defined by

$$\widehat{\beta}_t(q) = \exp^g(\hat{r}(q)\mathbf{n}_q)$$

is also Lipschitz. The restriction of this function to  $\tilde{A}_\delta \cap U$  agrees with  $\beta_t$ , completing the proof.  $\square$

For future reference, we note the following corollary.

**Corollary 1.3.34.** *Let  $\tilde{A}_\delta$  be the full-density subset of a  $\delta$ -flow set  $A_\delta$ , and let  $U$  be any set such that the generator flow with respect to some Riemannian metric  $\sigma$  is defined on all of  $U \cap A_\delta$ . Let  $\mathfrak{h}^n$  be the  $n$ -dimensional Hausdorff measure associated to  $\sigma$ . Then  $\beta_t(\tilde{A}_\delta \cap U)$  has full  $\mathfrak{h}^n$ -measure in  $\beta_t(A_\delta \cap U)$ .*

*Proof.* When  $U$  is contained in a sufficiently small open set, Lemma 1.3.33 tells us that  $\beta_t$  is the restriction of a Lipschitz function. Hence  $\beta_t(A_\delta \cap U) \setminus \beta_t(\tilde{A}_\delta \cap U)$  has  $\mathfrak{h}^n$ -measure zero, since  $A_\delta \setminus \tilde{A}_\delta$  has  $\mathfrak{h}^n$ -measure zero. Otherwise,  $U$  may be covered by countably many such small open sets since it is second-countable, giving the same conclusion.  $\square$

**Proposition 1.3.35.** *Let  $\mathcal{H}$  be a horizon in an  $(n+1)$ -dimensional spacetime  $(M, g)$  equipped with a Riemannian metric  $\sigma$  of the form*

$$\sigma(X, Y) = g(X, Y) + 2g(X, V)g(Y, V) \quad \text{for some timelike } g\text{-unit vector field } V.$$

Let  $\mathfrak{h}^n$  be the  $n$ -dimensional Hausdorff measure associated to  $\sigma$ , let  $\delta > 0$ , let  $A_\delta$  be the  $\delta$ -flow set, let  $\tilde{A}_\delta$  be the full-density subset of  $A_\delta$ , let  $t > 0$ , let  $\beta_t$  be the restriction to  $A_\delta$  of the generator flow with respect to  $\sigma$ , let  $U$  be a neighborhood of  $p$  which is open in  $M$ , suppose that  $\beta_t$  is defined on all of  $A_\delta \cap U$ , let  $N \subseteq U$  be a  $C^{1,1}$  hypersurface containing  $A_\delta$  and let  $\widehat{\beta}_t: N \rightarrow M$  be a locally Lipschitz function which agrees with  $\beta_t$  on  $\tilde{A}_\delta \cap U$ . Suppose that  $\theta_{\mathcal{A}I} \leq 0$  on all of  $\mathcal{H}_{\mathcal{A}I}$ .

Then every  $p \in A_\delta$  has a neighborhood  $Z$  which is open in  $A_\delta$  such that there is a measurable function  $\Psi$  with  $\Psi \geq 1$  almost everywhere such that if  $\varphi: A_\delta \rightarrow \mathbb{R}$  is  $\mathfrak{h}^n$ -integrable and supported in  $Z \cap \tilde{A}_\delta$  then

$$\int_{\tilde{A}_\delta} \varphi \Psi d\mathfrak{h}^n = \int_{\beta_t(\tilde{A}_\delta)} \varphi(\beta_t^{-1}(y)) d\mathfrak{h}^n(y).$$

Moreover, if  $\Psi = 1$  almost everywhere on  $Z$ , then  $\theta_{\mathcal{A}I} = 0$  almost everywhere on  $Z$ .

**Remark 1.3.36.** A sufficient condition for having  $\theta_{\mathcal{A}I} \leq 0$  is that the generators of  $\mathcal{H}$  are complete in the past direction, together with the null energy condition. (See Proposition 1.3.23.)

*Proof.* Let  $Z = A_\delta \cap U$ . By hypothesis  $\widehat{\beta}_t$  is Lipschitz on  $N$ . Theorem 3.1 of [12] tells us that

$$\int_N \varphi J(\widehat{\beta}_t) d\mathfrak{h}^n = \int_N \left( \sum_{x \in \widehat{\beta}_t^{-1}(y)} \varphi(x) \right) d\mathfrak{h}^n(y).$$

Here  $J(\widehat{\beta}_t)$  is the Jacobian determinant of  $\widehat{\beta}_t$  with respect to  $\sigma$  at points where  $\widehat{\beta}_t$  is differentiable. Since  $\widehat{\beta}_t$  is Lipschitz,  $J(\widehat{\beta}_t)$  is thus almost everywhere defined on  $N$ , which is sufficient for the integral to make sense.

Note that  $\varphi$  is zero outside of  $\tilde{A}_\delta$ , so

$$\sum_{x \in \widehat{\beta}_t^{-1}(y)} \varphi(x) = \sum_{x \in \widehat{\beta}_t^{-1}(y) \cap \tilde{A}_\delta} \varphi(x).$$

Since  $\beta_t$  is a bijection from  $A_\delta$  to  $\beta_t(A_\delta)$ , its restriction to  $\tilde{A}_\delta$  is injective. Its image is of course  $\beta_t(\tilde{A}_\delta)$ . This means that

$$\widehat{\beta}_t^{-1}(y) \cap \tilde{A}_\delta = \begin{cases} \{\beta_t^{-1}(y)\} & \text{if } y \in \beta_t(\tilde{A}_\delta), \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence

$$\int_{\tilde{A}_\delta} \varphi J(\widehat{\beta}_t) d\mathfrak{h}^n = \int_{\beta_t(\tilde{A}_\delta)} \varphi(\beta_t^{-1}(y)) d\mathfrak{h}^n(y).$$

To complete the proof of the theorem, we need to show that  $J(\widehat{\beta}_t) \geq 1$  almost everywhere on  $\tilde{A}_\delta \cap U$ , and that  $J(\widehat{\beta}_t) = 1$  almost everywhere on  $\tilde{A}_\delta \cap U$  only if



$\theta_{\mathcal{A}l} = 0$  almost everywhere on  $A_\delta \cap U$ , possibly after shrinking  $U$ . We can then choose  $\Psi$  to be  $J(\widehat{\beta}_t)$ . Since the argument for proving these statements is quite long, we prove them separately as Lemma 1.3.37.  $\square$

**Lemma 1.3.37.** *Fix  $t$ . Let  $p, U, A_\delta, \tilde{A}_\delta, N$  and  $\widehat{\beta}_t$  be as in Proposition 1.3.35. After possibly shrinking  $U$  to a smaller neighborhood of  $p$ , it holds that  $J(\widehat{\beta}_t) \geq 1$  almost everywhere (with respect to  $\mathfrak{h}^n$ ) on  $A_\delta \cap U$ . Moreover, if  $J(\widehat{\beta}_t) = 1$  almost everywhere on  $A_\delta \cap U$  then  $\theta_{\mathcal{A}l} = 0$  almost everywhere on  $A_\delta \cap U$ .*

*Proof.* **Proof part I(Construction of the set  $\widehat{B}$ ):** After possibly shrinking  $U$  to a smaller neighborhood of  $p$  and decomposing it as  $U = (a, b) \times \Sigma$  (with the curves  $(a, b) \times \{q\}$  timelike and the slices  $\{\tau\} \times \Sigma$  spacelike) where  $\Sigma$  is an open subset of  $\mathbb{R}^n$ , we may view  $\mathcal{H}$  as the graph of a semi-convex function  $f$ , as noted in Section 1.3.2.1. Similarly,  $N$  is the graph of a  $C^{1,1}$  function  $g$ . Let  $\mathcal{L}^n$  denote Lebesgue measure on  $\Sigma$ . By [13, Theorem 3.1.15], for each positive integer  $k$  there is a  $C^2$  function  $g_k: \Sigma \rightarrow \mathbb{R}$  such that

$$\mathcal{L}^n(\{x \in \Sigma \mid g_k(x) \neq g(x)\}) < 1/k.$$

Let  $\text{pr } \tilde{A}_\delta$  be the projection of  $\tilde{A}_\delta$  on  $\Sigma$ . Let  $B$  be the full-density subset of  $\text{pr } \tilde{A}_\delta$ . By Proposition A.4.7 the set  $B$  has full  $\mathcal{L}^n$ -measure in  $\text{pr } \tilde{A}_\delta$ . Letting

$$B_k = B \cap \{x \in \Sigma \mid g_k(x) = g(x)\}$$

we then have

$$\mathcal{L}^n(B \setminus B_k) < 1/k.$$

Once again, we discard low-density points: Let  $\tilde{B}_k$  be the full-density subset of  $B_k$ . Then  $\tilde{B}_k$  has full measure in  $B_k$  by Proposition A.4.7 so

$$\mathcal{L}^n(B \setminus \tilde{B}_k) < 1/k.$$

Let  $\Sigma_{Rad}$  denote the points of  $\Sigma$  where  $g$  is twice differentiable in the sense that it has second order expansions of the form

$$\begin{aligned} g(x) &= g(x_0) + dg(x_0)(x - x_0) + \frac{1}{2}D^2g(x_0)(x - x_0, x - x_0) + o(|x - x_0|^2), \\ dg(x) &= dg(x_0) + D^2g(x_0)(x - x_0, \cdot) + o(|x - x_0|). \end{aligned} \quad (1.4)$$

Rademacher's theorem tells us that since  $g$  is  $C^{1,1}$ , the set  $\Sigma_{Rad}$  has full measure in  $\Sigma$ . Defining

$$\begin{aligned} \widehat{B}_k &:= \tilde{B}_k \cap \Sigma_{Rad}, \\ \widehat{B} &:= \bigcup_{k \in \mathbb{N}} \widehat{B}_k = \Sigma_{Rad} \cap \bigcup_{k \in \mathbb{N}} \tilde{B}_k \end{aligned}$$

we then know that  $\widehat{B}$  has full  $\mathcal{L}^n$ -measure in  $\text{pr } \tilde{A}_\delta$ . Since  $g$  is Lipschitz, the graph of  $g$  over  $\widehat{B}$  has full  $\mathfrak{h}^n$ -measure in  $\tilde{A}_\delta \cap U$  by Proposition A.4.3. It is now sufficient to show that  $J(\widehat{\beta}_t)(p_0) \geq 1$  whenever  $p_0 = (g(x_0), x_0)$  is such that  $x_0 \in \widehat{B}$ .

Choose some  $x_0 \in \widehat{B}$ . Since  $\widehat{B} \subseteq \text{pr } \tilde{A}_\delta$ , the functions  $f$  and  $g$  agree at  $x_0$ , and  $x_0$  is an Alexandrov point of  $f$ . Hence we have the expansion

$$f(x) = f(x_0) + df(x_0)(x - x_0) + \frac{1}{2}D^2f(x_0)(x - x_0, x - x_0) + o(|x - x_0|^2).$$

Since  $x_0 \in \widehat{B}$ , we know that  $g$  is twice differentiable at  $x_0$  so that we have the expansions of equation (1.4). Moreover,  $x_0 \in B_i$  for some  $i \in \mathbb{N}$  so  $g_i(x_0) = g(x_0)$ . Fix this value of  $i$  for the remainder of the proof. Moreover, by definition of  $\widehat{B}$ , the point  $x_0$  is a full-density point of  $\text{pr } A_\delta$ , and  $g$  and  $f$  agree on  $\text{pr } A_\delta$ . Similarly,  $x_0$  is a full-density point of  $B_i$ , and  $g$  and  $g_i$  agree on  $B_i$ . This means that we can use Proposition A.4.12 to conclude that

$$df(x_0) = dg(x_0) = dg_i(x_0) \tag{1.5}$$

and

$$D^2g_i(x_0) = D^2g(x_0) = D^2f(x_0).$$

**Proof part II(Construction of the  $C^2$  null hypersurface  $\mathcal{H}_i$ ):** Let  $N_i$  denote the graph of  $g_i$  over  $\Sigma$ . Let  $\widehat{N}_i$  denote a  $C^2$  spacelike hypersurface in  $N_i$  containing  $p_0$ . Thus  $\widehat{N}_i$  has codimension 2 in the spacetime  $M$ . Let  $\mathbf{n}_i$  denote the past-directed  $\sigma$ -unit normal null vector field of  $\widehat{N}_i$  such that  $\mathbf{n}_i(p_0)$  is the  $\sigma$ -unit tangent of the (unique since  $p_0 \in A_\delta$ ) generator of  $\mathcal{H}$  passing through  $p_0$ . Note that  $\mathbf{n}_i$  is  $C^1$ . Let  $\mathcal{G}_i$  be the union of the geodesics starting from  $\widehat{N}_i$  with initial velocities given by  $\mathbf{n}_i$ . Let  $\gamma: [0, t] \rightarrow \mathcal{G}_i$  denote the curve  $s \mapsto \beta_s(p_0)$ . We wish to choose a subset of  $\mathcal{G}_i$  which is a  $C^2$  hypersurface containing  $\gamma$ . Define  $\exp: \Omega \rightarrow M$  by  $\exp(\tau, q) = \exp_q(\tau \mathbf{n}_i(q))$ , where  $\Omega \subseteq \mathbb{R} \times \widehat{N}_i$  is the largest subset on which  $\exp$  may be defined. Proposition A.3 of [8] says that if  $\exp_*$  is injective at  $(\tau, q)$  then there is an open neighborhood  $\mathcal{O}$  of  $(\tau, q)$  such that  $\exp(\mathcal{O})$  is a  $C^2$  submanifold of  $M$ . This, together with the fact that  $\exp$  is injective when restricted to  $[0, t] \times \{p_0\}$ , shows that some neighborhood of  $\gamma$  in  $\mathcal{G}_i$  is a  $C^2$  hypersurface in  $M$ . Hence we need to show that  $\exp_*$  is injective at  $(s, p_0)$  for each  $s \in [0, t]$ . Note that  $\widehat{N}_i$  is a  $C^2$  spacelike submanifold of  $M$  of codimension 2, and that  $\widehat{N}_i$  is *second order tangent* to  $\mathcal{H}$  at  $p_0$  in the sense of [8, Section 4.2] since  $D^2g_i(x_0) = D^2f(x_0)$ . By [8, Lemma 4.15] there can then be no focal point of  $\widehat{N}_i$  along  $\gamma$ . By [28, Proposition 10.30] this means that  $\exp_*$  is injective at  $(s, p_0)$  for all  $s \in [0, t]$ . As pointed out previously, [8, Proposition A.3] then tells us that some open neighborhood of  $\gamma$  in  $\mathcal{G}_i$  is a  $C^2$  submanifold. Since  $\exp_*$  is injective at each point on  $\gamma$ , it is injective in a neighborhood of each

such point. Since  $\gamma([0, t])$  is compact, finitely many such neighborhoods suffice to cover  $\gamma$ . Hence there is a neighborhood of  $\gamma$  where  $\mathcal{G}_i$  is  $C^2$  and  $\exp_*$  is injective. Denote this neighborhood by  $\mathcal{H}_i$ .

By an application of Proposition B.2.1 we see that  $\mathcal{H}_i$  is a null hypersurface.

**Proof part III(Definition of a map  $\widehat{\beta}_t^i : \mathcal{H}_i \rightarrow M$ ):** By definition of  $\mathcal{H}_i$ , the vector field  $\mathbf{n}_i$  (where defined) is tangent to  $\mathcal{H}_i$ . Since  $\mathcal{H}_i$  is a null hypersurface, it has a unique  $\sigma$ -unit normal null vector field, which must then be an extension of  $\mathbf{n}_i$ . Call this extension  $\mathbf{n}_i$  as well.

Note that  $\mathcal{H}_i$  contains both  $\widehat{N}_i$  and the generator passing through  $p_0$ , these being submanifolds of  $N_i$  transverse to each other. Hence the first and second derivatives of the graphing functions of  $\mathcal{H}_i$  and  $N_i$  must agree at  $p_0$ .

Define the map  $\widehat{\beta}_t^i : \mathcal{H}_i \rightarrow M$  by

$$\widehat{\beta}_t^i(q) = \exp^g(r(q)\mathbf{n}_i)$$

where  $r(q)$  is the unique nonnegative real number such that the  $\sigma$ -distance from  $q$  to  $\exp^g(r(q)\mathbf{n}_i)$  along the  $g$ -geodesic  $\exp^g(\tau\mathbf{n}_i)$  is precisely  $t$ . Then by definition

$$\widehat{\beta}_t^i(p_0) = \widehat{\beta}_t(p_0).$$

Note that  $\widehat{\beta}_t$  and  $\widehat{\beta}_t^i$  are defined by the formula  $\exp^g(r(q)\mathbf{k})$  where  $\mathbf{k}$  is the normal vector field of  $N$  and  $\mathcal{H}_i$ , respectively. The derivative of  $\exp^g(r(q)\mathbf{k})$  is determined by the first derivatives of  $r$  and  $\mathbf{k}$ . These in turn are determined by the second derivatives of the graphing functions of  $N$  and  $\mathcal{H}_i$ . Since  $dg(p_0) = dg_i(p_0)$  and  $D^2g(p_0) = D^2g_i(p_0)$  this means that the tangent maps of  $\widehat{\beta}_t^i$  and  $\widehat{\beta}_t$  agree at  $p_0$ . Hence

$$J(\widehat{\beta}_t^i)(p_0) = J(\widehat{\beta}_t)(p_0).$$

We have now reduced the problem to showing that  $J(\widehat{\beta}_t^i)(p_0) \geq 1$ .

**Proof part IV(Computation of  $J(\widehat{\beta}_t^i)(p_0)$ ):** Let  $b_{\mathcal{H}_i}$  be the one-parameter family of Weingarten maps (defined in Section 1.3.1.1) along the generator of  $\mathcal{H}_i$  through  $p_0$  with its affine parametrization, and let  $b'_{\mathcal{H}_i}$  denote the covariant derivative of  $b$  along this generator. Equation (1.1) in Section 1.3.1.1 tells us that  $b_{\mathcal{H}_i}$  satisfies the equation

$$b' + b^2 + \widetilde{R} = 0. \tag{1.6}$$

Recall from Remark 1.3.22 that  $\mathcal{H}$  has a null Weingarten map  $b_{\mathcal{A}l}$ , defined in terms of Alexandrov derivatives, on all points to the past of  $p_0$  on the generator of  $\mathcal{H}$  through  $p_0$ , and that this map also satisfies equation (1.6). Since the null Weingarten map of a null hypersurface can be expressed in the first two derivatives of a graphing function, and  $\mathcal{H}$  shares these derivatives with  $N_i$  which in turns shares

them with  $\mathcal{H}_i$ . Hence  $b_{\mathcal{H}_i}(p_0) = b_{\mathcal{A}I}(p_0)$ . By the uniqueness of solutions to the ordinary differential equation (1.6), these two maps must agree on all of the past of  $p_0$  along the generator through  $p_0$ . Let  $\theta_{\mathcal{H}_i}$  denote the null mean curvature of  $\mathcal{H}_i$ , as defined in Section 1.3.1.1. Then we have

$$\theta_{\mathcal{H}_i} = \text{tr } b_{\mathcal{H}_i} = \text{tr } b_{\mathcal{A}I} = \theta_{\mathcal{A}I}.$$

Lemma 1.3.4 implies that

$$J(\widehat{\beta}_t)^i(p_0) = \exp\left(-\int_0^t \theta_{\mathcal{H}_i}(\widehat{\beta}_s^i(p_0)) ds\right).$$

Since  $J(\widehat{\beta}_t)^i(p_0) = J(\widehat{\beta}_t)(p_0)$  and  $\theta_{\mathcal{A}I} = \theta_{\mathcal{H}_i}$  along the curve  $s \mapsto \widehat{\beta}_s^i(p_0) = \widehat{\beta}_s(p_0)$ , we then know that

$$J(\widehat{\beta}_t)(p_0) = \exp\left(-\int_0^t \theta_{\mathcal{A}I}(\widehat{\beta}_s(p_0)) ds\right).$$

Recall that  $g(\widehat{B})$  has full  $\mathfrak{h}^n$ -measure in  $N \cap U$  and that  $p_0$  was an arbitrary point of  $g(\widehat{B})$ . Since we have assumed that  $\theta_{\mathcal{A}I} \leq 0$ , we can conclude that  $J(\widehat{\beta}_t) \geq 1$  almost everywhere on the neighborhood  $U$  of  $p$ . This completes the proof of the first part of the lemma.

To prove the last part of the lemma, some measure theoretical technicalities remain.

**Claim V:**  $J(\widehat{\beta}_t) = 1$  almost everywhere on  $\tilde{A}_\delta \cap U$  only if  $\theta_{\mathcal{A}I} = 0$  almost everywhere on  $\tilde{A}_\delta \cap U$

After possibly shrinking  $U$ , we may choose  $C^1$  coordinates on  $N \cap U \cong (a, b) \times \Sigma$  in which

$$\widehat{\beta}_t((q, s)) = (q, s + t) \text{ for all } q \in \Sigma \text{ and } s, t \text{ such that } s \in (a, b) \text{ and } s + t \in (a, b).$$

Suppose that  $J(\widehat{\beta}_t) = 1$  almost everywhere on  $\tilde{A}_\delta \cap U$  with respect to  $\mathfrak{h}^n$ . Then in particular (since  $g(\widehat{B})$  has full measure in  $N \cap U$ ) it holds that  $J(\widehat{\beta}_t) = 1$  almost everywhere on  $\tilde{A}_\delta \cap U \cap g(\widehat{B})$ . We have seen that on this set, with coordinates  $(q, s) \in (a, b) \times \Sigma$ ,

$$J(\widehat{\beta}_t)((q, s)) = \exp\left(-\int_0^t \theta_{\mathcal{A}I}(\widehat{\beta}_s((q, s))) ds\right).$$

If  $J(\widehat{\beta}_t) = 1$  almost everywhere, then it holds for  $\mathfrak{h}^n$ -almost every pair  $(q, s)$  that  $\theta_{\mathcal{A}I}(q, s') = 0$  for almost all  $s' > s$  with respect to Lebesgue measure. By shrinking  $U$  further, we may assume that it holds for  $\mathfrak{h}^n$ -almost every  $q \in \Sigma$  that  $\theta_{\mathcal{A}I}(q, s) = 0$  for almost all  $s > 0$  with respect to Lebesgue measure. Note that the measure  $\mathfrak{h}^n$  differs from the Lebesgue measure in coordinates of  $(a, b) \times \Sigma$  only by pointwise scaling by

a  $C^1$  function. Note also that Lebesgue measure in coordinates on  $(a, b) \times \Sigma$  agrees with the product of Lebesgue measure on  $\Sigma$  and Lebesgue measure on  $(a, b)$  (on sets which are measurable in both measures). Hence Fubini's theorem (see [13, Theorem 2.6.2]) implies that

$$\begin{aligned} \mathcal{L}^n(\{(s, q) \in (a, b) \times \Sigma \mid \theta_{\mathcal{A}I}(q, s) \neq 0\}) &= \int_{\Sigma} \mathcal{L}^1(\{s \in (a, b) \mid \theta_{\mathcal{A}I}(q, s) \neq 0\}) d\mathcal{L}^{n-1}(q) \\ &= \int_{\Sigma} 0 d\mathcal{L}^{n-1}(q) = 0. \end{aligned}$$

This shows that  $\theta_{\mathcal{A}I} = 0$  on almost all of  $U \cap A_{\delta}$  with respect to  $\mathfrak{h}^n$ , completing the proof.  $\square$

**Proposition 1.3.38.** *Let  $\mathcal{H}$  be a horizon in an  $(n+1)$ -dimensional spacetime  $M$  equipped with a Riemannian metric  $\sigma$  of the form*

$$\sigma(X, Y) = g(X, Y) + 2g(X, V)g(Y, V) \quad \text{for some timelike } g\text{-unit vector field } V.$$

*Let  $\delta > 0$  and let  $A_{\delta}$  be the  $\delta$ -flow set of  $\mathcal{H}$ . Let  $\mathfrak{h}^n$  denote the  $n$ -dimensional Hausdorff measure induced by  $\sigma$ . Let  $\hat{\mathcal{H}}$  be a past null geodesically ruled open subset of  $\mathcal{H}$ . Suppose that  $\theta_{\mathcal{A}I} \leq 0$  on all of  $\mathcal{H}_{\mathcal{A}I} \cap \hat{\mathcal{H}}$ . Let  $t > 0$  be such that the generator flow  $\beta_t$  is defined on all of  $\hat{\mathcal{H}} \cap A_{\delta}$ .*

*Then*

$$\mathfrak{h}^n(\hat{\mathcal{H}} \cap A_{\delta}) = \mathfrak{h}^n(\beta_t(\hat{\mathcal{H}} \cap A_{\delta})).$$

*Moreover,  $\theta_{\mathcal{A}I} = 0$  almost everywhere on  $\hat{\mathcal{H}} \cap A_{\delta}$ .*

*Proof.* Note that the generators of  $\hat{\mathcal{H}}$  are the intersections of the generators of  $\mathcal{H}$  with  $\hat{\mathcal{H}}$  since  $\hat{\mathcal{H}}$  is an open subset of the horizon  $\mathcal{H}$ . For each  $p$ , let  $Z_p$  and  $\Psi_p$  be the neighborhoods and functions given by Proposition 1.3.35. Since  $A_{\delta}$  is second countable, a countable number of neighborhoods  $Z_1, Z_2, \dots$  suffice to cover  $A_{\delta}$ . For each  $i \geq 1$ , let

$$Y_i = Z_i \setminus \bigcup_{1 \leq j < i} Z_j.$$

Then

- each  $Y_i$  is measurable,
- $Y_i \subseteq Z_i$  for each  $i$ ,
- the  $Y_i$  are pairwise disjoint,
- $\bigcup_{i \geq 1} Y_i = \bigcup_{i \geq 1} Z_i \supseteq A_{\delta}$ .

For each  $i \geq 1$ , let  $\varphi_i$  be the indicator function of  $Y_i \cap \hat{\mathcal{H}} \cap \tilde{A}_\delta$ . Then Proposition 1.3.35 says that

$$\int_{\tilde{A}_\delta} \varphi_i \Psi_i d\mathfrak{h}^n = \int_{\beta_t(\tilde{A}_\delta)} \varphi_i(\beta_t^{-1}(y)) d\mathfrak{h}^n(y)$$

for each  $i \geq 1$ . Since each  $\varphi_i$  is zero outside of  $\hat{\mathcal{H}}$ , this means that

$$\int_{\hat{\mathcal{H}} \cap \tilde{A}_\delta} \varphi_i \Psi_i d\mathfrak{h}^n = \int_{\beta_t(\hat{\mathcal{H}} \cap \tilde{A}_\delta)} \varphi_i(\beta_t^{-1}(y)) d\mathfrak{h}^n(y)$$

Taking a sum over  $i$ , we see that

$$\int_{\hat{\mathcal{H}} \cap \tilde{A}_\delta} \sum_{i \geq 1} \varphi_i \Psi_i d\mathfrak{h}^n = \int_{\beta_t(\hat{\mathcal{H}} \cap \tilde{A}_\delta)} \sum_{i \geq 1} \varphi_i(\beta_t^{-1}(y)) d\mathfrak{h}^n(y).$$

Since precisely one of the functions  $\varphi_i$  is nonzero at any point  $p \in \hat{\mathcal{H}} \cap \tilde{A}_\delta$ , and takes the value 1 there, we have  $\sum_{i \geq 1} \varphi_i \Psi_i \geq 1$  almost everywhere on  $\hat{\mathcal{H}} \cap \tilde{A}_\delta$ . Moreover, we have  $\sum_{i \geq 1} \varphi_i(\beta_t^{-1}(y)) = 1$  almost everywhere on  $\beta_t(\hat{\mathcal{H}} \cap \tilde{A}_\delta)$ . Hence

$$\mathfrak{h}^n(\hat{\mathcal{H}} \cap \tilde{A}_\delta) \leq \int_{\hat{\mathcal{H}} \cap \tilde{A}_\delta} \sum_{i \geq 1} \varphi_i \Psi_i d\mathfrak{h}^n = \mathfrak{h}^n(\beta_t(\hat{\mathcal{H}} \cap \tilde{A}_\delta)).$$

Since  $\tilde{A}_\delta$  has full measure in  $A_\delta$  and  $\beta_t(\tilde{A}_\delta)$  has full measure in  $\beta_t(A_\delta)$  by Corollary 1.3.34, this means that

$$\mathfrak{h}^n(\hat{\mathcal{H}} \cap A_\delta) \leq \mathfrak{h}^n(\beta_t(\hat{\mathcal{H}} \cap A_\delta)).$$

However,  $\beta_t(\hat{\mathcal{H}} \cap A_\delta) \subseteq \hat{\mathcal{H}} \cap A_\delta$  since the generators of  $\hat{\mathcal{H}}$  agree with the generators of  $\mathcal{H}$ , so we also know that

$$\mathfrak{h}^n(\hat{\mathcal{H}} \cap A_\delta) \geq \mathfrak{h}^n(\beta_t(\hat{\mathcal{H}} \cap A_\delta))$$

by additivity of the measure. Hence equality must hold, and the proof of the first statement is complete.

Equality can hold only if  $\sum_{i \geq 1} \varphi_i \Psi_i = 1$  almost everywhere on  $\hat{\mathcal{H}} \cap A_\delta$ . This means that each function  $\Psi_i$  must be equal to 1 almost everywhere on  $Y_i \cap \hat{\mathcal{H}} \cap A_\delta$ . By Proposition 1.3.35 this implies that  $\theta_{\mathcal{A}l} = 0$  almost everywhere on  $Y_i \cap \hat{\mathcal{H}} \cap A_\delta$ . Since these sets cover  $\hat{\mathcal{H}} \cap A_\delta$ , we have shown that  $\theta_{\mathcal{A}l} = 0$  almost everywhere on  $\hat{\mathcal{H}} \cap A_\delta$  with respect to the measure  $\mathfrak{h}^n$ . This completes the proof.  $\square$

#### 1.3.5.4 Smoothness from area-preserving generator flow

**Lemma 1.3.39.** *Let  $\mathcal{H}$  be a horizon in a spacetime of dimension  $n+1$  equipped with a Riemannian metric  $\sigma$  and the corresponding Hausdorff measure  $\mathfrak{h}^n$ . Let  $\Omega$  be an open subset of  $\mathcal{H}$ . Let  $A_\delta$  denote the  $\delta$ -flow set of  $\mathcal{H}$  with respect to  $\sigma$ . Suppose that*

$$\mathfrak{h}^n(\Omega \cap A_\delta) = \mathfrak{h}^n(\beta_t(\Omega \cap A_\delta))$$

for all  $t > 0$  and all  $\delta > 0$ , and that  $\mathfrak{h}^n(\Omega) < \infty$ .

Then the following two statements hold.

1. There is a dense subset of  $\Omega$  consisting of points on maximal null geodesics contained in  $\Omega$ .
2. No generator of  $\mathcal{H}$  has any endpoint on  $\Omega$ .

*Proof.* Let  $A_\delta$  denote the  $\delta$ -flow set of  $\mathcal{H}$  with respect to  $\sigma$ . Recall that the total flow set of  $\mathcal{H}$  is the set

$$A_0 = \bigcup_{\delta > 0} A_\delta.$$

Note that if  $\delta < \delta'$  then  $A_\delta \supseteq A_{\delta'}$ . Hence

$$A_0 = \bigcup_{\delta > 0} A_\delta = \bigcup_{k \in \mathbb{Z}^+} A_{1/k}.$$

Since the family  $\Omega \cap A_{1/k}$  is increasing,

$$\mathfrak{h}^n(\beta_t(\Omega \cap A_0)) = \lim_{k \rightarrow \infty} \mathfrak{h}^n(\beta_t(\Omega \cap A_{1/k})) = \lim_{k \rightarrow \infty} \mathfrak{h}^n(\Omega \cap A_{1/k}) = \mathfrak{h}^n(\Omega \cap A_0).$$

The limits are finite, since by hypothesis  $\mathfrak{h}^n(\Omega \cap A_{1/k})$  is uniformly bounded with respect to  $k$  by  $\mathfrak{h}^n(\Omega)$ .

Introduce the sets  $\mathfrak{C}$  and  $\mathfrak{D}$  defined by

$$\mathfrak{C} = \bigcap_{t \in \mathbb{Z}^+} \beta_t(\Omega \cap A_0),$$

$$\mathfrak{D} = \{p \in \Omega \mid \text{there is a unique generator through } p, \\ \text{and this generator has no future endpoint}\}.$$

We first show that if  $p \in \mathfrak{C}$  then the generator  $\Gamma_p$  through  $p$  is a maximal geodesic contained in  $\Omega$ . By choice of  $\Omega$ , the part of the generator to the past of  $p$  belongs to  $\Omega$ , and is maximal in the past direction. Parameterize  $\Gamma_p$  by an affine parameter such that  $\Gamma_p(0) = p$ . Suppose that the maximal future extension of the generator were to leave  $\Omega$  at some point  $q = \Gamma_p(s)$ . Since  $\Gamma_p$  is smooth and the interval  $[0, s]$  is compact, the curve segment  $\Gamma_p([0, s])$  has finite length in the Riemannian metric  $\sigma$ . This means that  $p \notin \beta_t(\Omega \cap A_0)$  whenever  $t$  is greater than this length, contradicting the assumption that  $p \in \mathfrak{C}$ . This shows that the set  $\mathfrak{C}$  satisfies the conditions for the first statement in the conclusion, so that it is sufficient to show that  $\mathfrak{C}$  is dense to complete the proof of that statement.

Note that the fact that generators through points of  $\mathfrak{C}$  are maximal means that they can have no endpoints. Hence  $\mathfrak{C} \subseteq \mathfrak{D}$ . Since  $(\beta_t(\Omega \cap A_0))_{t=1}^\infty$  is a countable decreasing family of sets of equal measure it holds that  $\mathfrak{h}^n(\mathfrak{C}) = \mathfrak{h}^n(\Omega \cap A_0)$ . Since

$A_0 \subseteq \mathcal{H}$ , and  $\mathfrak{h}^n(\mathcal{H} \setminus A_0) = 0$  by Proposition 1.3.29, this means that  $\mathfrak{h}^n(\Omega) = \mathfrak{h}^n(\Omega \cap A_0)$  so that

$$\mathfrak{h}^n(\mathfrak{C}) = \mathfrak{h}^n(\Omega).$$

In particular,  $\mathfrak{C}$  is dense in  $\Omega$ . Since  $\mathfrak{C} \subseteq \mathfrak{D}$ , it follows that  $\mathfrak{D}$  is also dense in  $\Omega$ .

We will now show that  $\mathfrak{D}$  is closed in  $\Omega$ . Suppose that a sequence  $(p_k)_{k \in \mathbb{N}}$  in  $\mathfrak{D}$  converges to  $p \in \Omega$ . Let  $X_k$  denote the (unique, by definition of  $\mathfrak{D}$ ) future-directed  $\sigma$ -unit tangent of a generator at  $p_k$ . The  $\sigma$ -unit tangent bundle over the compact countable set  $\{p\} \cup \{p_1, p_2, \dots\}$  is compact, so by passing to a subsequence we may assume that the  $X_k$  converge to some unit vector  $X$  at  $p$ . By [8, Lemma 6.4] the space of past-directed  $\sigma$ -unit generator tangents is closed in the unit tangent bundle, so we know that  $X$  is tangent to a generator. Let  $\gamma$  denote the inextendible geodesic with initial velocity  $X$ , and let  $\gamma_k$  denote the inextendible geodesic with initial velocity  $X_k$ . By definition of  $\mathfrak{D}$ , each geodesic  $\gamma_k$  avoids the open set  $I^+(\mathcal{H})$ . By continuous dependence on initial conditions for ordinary differential equations,  $\gamma$  must also avoid the open set  $I^+(\mathcal{H})$ . Suppose to get a contradiction that  $\gamma$  leaves  $\mathcal{H}$  at some point  $p$ . Choose coordinates around  $p$  of the form  $(-a, a) \times \Sigma$ , such that each curve  $(-a, a) \times \{q\}$  is timelike and each slice  $\{t\} \times \Sigma$  is spacelike. Let  $\Sigma'$  denote the projection of  $\text{im } \gamma \cap ((-a, a) \times \Sigma)$  on  $\Sigma$ . Recall that  $\mathcal{H}$  is an achronal hypersurface, so by possibly shrinking  $\Sigma$  we may represent  $\mathcal{H} \cap ((-a, a) \times \Sigma')$  as the graph of a function  $f_{\mathcal{H}}: \Sigma' \rightarrow (-a, a)$ . Let  $f_{\gamma}: \Sigma' \rightarrow (-a, a)$  be the function the graph of which is the image of  $\gamma$  (on both sides of the point  $p$ ). Recall that  $\gamma$  is a null curve, so if  $f_{\mathcal{H}}(x) > f_{\gamma}(x)$  at some point  $x \in \Sigma'$  then there is a timelike curve from  $(f_{\mathcal{H}}(x), x)$  to  $p$ , contradicting achronality of  $\mathcal{H}$ . If  $f_{\mathcal{H}}(x) < f_{\gamma}(x)$  at some point  $x \in \Sigma'$  then  $\gamma$  intersects  $I^+(\mathcal{H})$  which we saw earlier is impossible. Hence  $\gamma$  cannot leave  $\mathcal{H}$  to the future, and so there is a generator through  $p$  without future endpoint. Moreover,  $p$  is then an interior point of a generator so this generator is unique. Hence  $p \in \mathfrak{D}$  and we have shown that  $\mathfrak{D}$  is closed in  $\Omega$ .

We have now shown that  $\mathfrak{D}$  is a closed dense subset of  $\Omega$ . Hence  $\mathfrak{D} = \Omega$ . Since no point in  $\mathfrak{D}$  lies on a generator with a future endpoint, no point in  $\mathfrak{D}$  can be a future endpoint of a generator. This shows that no generator of  $\mathcal{H}$  can have a future endpoint on  $\Omega$ . Recall that no generator has any past endpoint either, since  $\mathcal{H}$  is a horizon. This completes the proof.  $\square$

The condition of  $\mathcal{H}$  containing no endpoints is very strong, as the next theorem (see [8, Theorem 6.18]) illustrates.

**Theorem 1.3.40.** *Suppose that  $\Omega$  is an open subset of a horizon  $\mathcal{H}$  in a spacetime  $M$ , such that  $\Omega$  contains no endpoints of generators of  $\mathcal{H}$ . Suppose moreover that  $\theta_{\mathcal{A}I} = 0$  almost everywhere with respect to the  $n$ -dimensional Hausdorff measure  $\mathfrak{h}^n$  induced by a Riemannian metric  $\sigma$  on  $M$ . Then  $\Omega$  is a smooth submanifold of  $M$ .*



We are now in a position to prove the main theorem of this section. It was shown in [6, Section 4] that not all compact horizons are smooth. Our theorem shows that the additional hypothesis of the null energy condition is sufficient to guarantee smoothness.

**Theorem 1.3.41.** *Let  $M$  be a spacetime of dimension  $n+1$  satisfying the null energy condition. Let  $S \subset M$  be an achronal set with  $\text{edge}(S) = \emptyset$ . Let  $\hat{\mathcal{H}}$  be an open subset of  $H^+(S)$  with compact closure. Suppose that  $\hat{\mathcal{H}}$  is past null geodesically ruled. Then  $\hat{\mathcal{H}}$  is a smooth totally geodesic null hypersurface.*

*Proof.* First note that Corollary 1.3.14 tells us that  $H^+(S)$  is a horizon in the sense of Definition 1.3.7. Further note that  $\hat{\mathcal{H}}$  is a Lipschitz hypersurface, since it is an open subset of the Lipschitz hypersurface  $H^+(S)$ . It is also assumed to be past null geodesically ruled, so the generators of  $\hat{\mathcal{H}}$  are the intersection of the generators of  $H^+(S)$  with  $\hat{\mathcal{H}}$ . Note that Alexandrov points of  $H^+(S)$  are Alexandrov points of  $\hat{\mathcal{H}}$ . Each generator of  $\hat{\mathcal{H}}$  is a part of a null geodesic contained in  $H^+(S)$ . By definition, each generator of  $\hat{\mathcal{H}}$  is completely contained in, and hence totally past imprisoned in, the compact set  $\hat{\mathcal{H}}$ . This means that the generator flow  $\beta_t$  is defined for all  $t$  on all of the part of the total flow set of  $H^+(S)$  which lies in  $\hat{\mathcal{H}}$ . Moreover, Lemma B.1.1 shows that each generator of  $\hat{\mathcal{H}}$  is complete in the past direction. Since the null energy condition holds and we have shown that the generators of  $\hat{\mathcal{H}}$  are complete to the past, Proposition 1.3.23 tells us that  $\theta_{\mathcal{A}I} \leq 0$  at all Alexandrov points of  $\hat{\mathcal{H}}$ . Let  $V$  be an arbitrary timelike vector field on  $M$ , introduce the Riemannian metric  $\sigma$  on  $M$  defined by

$$\sigma(X, Y) = g(X, Y) + 2g(X, V)g(Y, V),$$

and let  $\mathfrak{h}^n$  be the corresponding  $n$ -dimensional Hausdorff measure. By Proposition A.4.4, the set  $\hat{\mathcal{H}}$  and all its measurable subsets have finite  $\mathfrak{h}^n$ -measure since  $\hat{\mathcal{H}}$  is compact. For each  $\delta > 0$ , let  $A_\delta$  denote the  $\delta$ -flow set of  $H^+(S)$ . By Proposition 1.3.38 we know that

$$\mathfrak{h}^n(\hat{\mathcal{H}} \cap A_\delta) = \mathfrak{h}^n(\beta_t(\hat{\mathcal{H}} \cap A_\delta))$$

and

$$\theta_{\mathcal{A}I} = 0 \text{ almost everywhere on } \hat{\mathcal{H}} \cap A_\delta$$

for all  $t > 0$  and all  $\delta > 0$ . Lemma 1.3.39 then tells us that no generator of  $H^+(S)$  has any endpoint on  $\hat{\mathcal{H}}$ . Moreover, since the total flow set  $A_0 = \bigcup_{\delta > 0} A_\delta$  has full  $\mathfrak{h}^n$ -measure by Proposition 1.3.29 we see that  $\theta_{\mathcal{A}I} = 0$  almost everywhere on  $\hat{\mathcal{H}}$ . Theorem 1.3.40 then says that  $\hat{\mathcal{H}}$  is a smooth submanifold of  $M$ . Since  $\hat{\mathcal{H}}$  is an open subset of  $H^+(S)$  and the tangent space of  $H^+(S)$  is a null hyperplane whenever it exists,  $\hat{\mathcal{H}}$  is a null hypersurface.

Let  $K$  be the tangent vector field of the generators of  $\widehat{\mathcal{H}}$ , scaled to an affine parametrization. Since  $\widehat{\mathcal{H}}$  is smooth, its null mean curvature  $\theta$  with respect to  $K$  is a smooth function and its sign agrees with that of the Alexandrov null mean curvature  $\theta_{\mathcal{A}l}$ . We saw previously that  $\theta_{\mathcal{A}l} = 0$  almost everywhere, so by continuity  $\theta = 0$  everywhere. Let  $b$  denote the null Weingarten map with respect to  $K$ , and let  $S = b - \frac{\theta}{n-2}$ . Since  $S$  is self-adjoint,  $\text{tr}(S^2) \geq 0$ . Since  $\theta = 0$  and  $\text{Ric}(K, K) \geq 0$  everywhere, equation (1.2) tells us that  $\text{tr}(S^2) = 0$ . Since  $S$  is self-adjoint this implies that  $S = 0$ . Hence  $b = 0$  everywhere, so that  $\widehat{\mathcal{H}}$  has vanishing null second fundamental form. Theorem 1.3.1 then implies that  $\widehat{\mathcal{H}}$  is totally geodesic, completing the proof.  $\square$

The following corollary is immediate.

**Corollary 1.3.42.** *Let  $M$  be a spacetime satisfying the null energy condition. Let  $S \subset M$  be an achronal set with  $\text{edge}(S) = \emptyset$ . Suppose that  $H^+(S)$  is compact. Then  $H^+(S)$  is a smooth totally geodesic null hypersurface.*

*Proof.* Let  $\widehat{\mathcal{H}} = H^+(S)$  and apply Theorem 1.3.41.  $\square$

### 1.3.6 Cobordisms satisfying energy conditions are trivial

In this section, we will apply Theorem 1.3.41 to prove Tipler's theorem.

#### 1.3.6.1 Cylinder extensions

We begin by describing a construction embedding a Lorentzian cobordism  $M$  into a manifold without boundary  $\Xi(M)$ . Informally, we will glue two cylinders onto  $M$ . This construction is a technicality needed only to apply theorems about manifolds without boundary to a cobordism.

**Definition 1.3.43.** Let  $M$  be a Lorentzian cobordism between  $S_1$  and  $S_2$ . Let

$$M_i = [0, \infty) \times S_i.$$

The *cylinder extension* of  $M$  is the spacetime without boundary

$$\Xi(M) = M_1 \sqcup M \sqcup M_2 / \sim$$

where  $\sim$  is the smallest equivalence relation such that

$$(0, p) \sim p \text{ for all } p \in S_i \text{ and } (0, p) \in M_i.$$

The metric of  $\Xi(M)$  is taken to be an arbitrary smooth extension of the metric on  $M$ .

**Remark 1.3.44.** Note that adding cylinders in this manner does not affect the causal relations of points in the original manifold  $M$ , since the time-orientation of the boundary does not allow causal curves to pass from  $M_2$  to  $M$  or from  $M$  to  $M_1$ .

**Lemma 1.3.45.** *Let  $M$  be a Lorentzian cobordism between  $S_1$  and  $S_2$ . Then  $S_1$  is a closed and achronal subset of  $\Xi(M)$ .*

*Proof.* Let  $M_i$  denote the open submanifold  $(0, \infty) \times S_i$  of  $\Xi(M)$  and let  $\mathring{M}$  denote the interior of  $M$ . The set  $S_1$  is a closed subset of  $\Xi(M)$  by the definition of the quotient topology on  $\Xi(M)$ , since its inverse image under the projection map  $M_1 \sqcup M \sqcup M_2 \rightarrow \Xi(M)$  is the closed set  $\{0\} \times S_1 \sqcup S_1$ .

To see that  $S_1$  is achronal, suppose that  $\gamma: [a, b] \rightarrow \Xi(M)$  is a future-directed timelike curve (with  $a < b$ ) such that  $\gamma(a), \gamma(b) \in S_1$ . Since  $S_1$  is closed, so is the set  $\gamma^{-1}(S_1)$ . If  $\gamma^{-1}(S_1)$  were dense in  $[a, b]$ , then by continuity  $\gamma$  would be completely contained in  $S_1$ . This is impossible since  $S_1$  is spacelike and  $\gamma$  timelike. Hence there is some maximal open interval  $J \subseteq (a, b)$  such that  $\gamma(J) \cap S_1 = \emptyset$ . By replacing  $a$  and  $b$  with  $\inf(J)$  and  $\sup(J)$ , we see that we may without loss of generality assume that  $\gamma(t) \notin S_1$  for  $t \in (a, b)$ . Since  $\dot{\gamma}(a)$  is future-directed, it must be inward-directed as a vector at the boundary  $S_1$  of  $M$ . Hence  $\gamma(a + \epsilon) \notin M_1$  for all sufficiently small  $\epsilon > 0$ . Since the boundary of  $M_1$  in  $\Xi(M)$  is  $S_1$ , continuity of  $\gamma$  implies that  $\gamma(t) \notin M_1$  for all  $t \in (a, b)$ . However,  $\dot{\gamma}(b)$  is also future-directed and hence inward-directed with respect to  $M$ , and so  $\gamma(b - \epsilon) \in M_1$  for all sufficiently small  $\epsilon > 0$ . This is a contradiction, so  $\gamma$  cannot intersect  $S_1$  more than once. Hence  $S_1$  is achronal.  $\square$

### 1.3.6.2 Tipler's theorem without smoothness hypothesis

We will prove the generalization of Tipler's theorem, suggested in [4], to causally compact Lorentzian pseudocobordisms.

**Theorem 1.3.46.** *Let  $n \geq 2$ . Let  $S_1, S_2$  be  $n$ -dimensional manifolds and let  $(M, g)$  be a causally compact Lorentzian pseudocobordism between  $S_1$  and  $S_2$  which satisfies the lightlike convergence condition and the lightlike generic condition.*

*Then  $M$  is globally hyperbolic. In particular  $M \cong S_1 \times [0, 1]$  so that  $S_1$  and  $S_2$  are diffeomorphic.*

*Proof.* We will work with the cylinder extension  $\Xi(M)$  of  $M$ , since we need a manifold without boundary. Let  $M_i$  for  $i \in \{1, 2\}$  denote the embedded submanifolds  $(0, \infty) \times S_i$  of  $\Xi(M)$ . Then  $\Xi(M)$  is the union of  $M_1, S_1, \mathring{M}, S_2$  and  $M_2$ , and these sets are disjoint. Consider the future Cauchy horizon  $H^+(S_1)$  of  $S_1$  in  $\Xi(M)$ . If it is empty, then  $S_1$  would be a Cauchy surface for  $M \cup M_2$  and hence also for  $M$ .

This would mean that  $M$  is globally hyperbolic. Hence it is sufficient to show that  $H^+(S_1)$  is empty. Suppose for contradiction that this is not the case.

**Claim I:  $H^+(S_1)$  is a horizon**

$S_1$  is a smooth hypersurface in  $\Xi(M)$ . Lemma 1.3.45 tells us that  $S_1$  is closed and achronal. Hence we can use Lemma 1.3.15 and Corollary 1.3.14 to conclude that  $H^+(S_1)$  is a horizon in the sense of Definition 1.3.7.

Choose a point  $p \in I^+(H^+(S_1), M)$ . To see that such a point exists, note that  $I^+(q, M)$  is nonempty if  $q \notin S_2$ , and that  $H^+(S_1) \setminus S_2$  is nonempty since  $H^+(S_1)$  is past null geodesically ruled and hence cannot be completely contained in a space-like hypersurface. With this choice,  $H^+(S_1) \cap I^-(p, M)$  is nonempty.

**Claim II: Every generator of  $H^+(S_1)$  which intersects  $I^-(p, M)$  stays in  $I^-(p, M)$  when followed to the past**

Let  $\gamma$  be a generator of  $H^+(S_1)$ . We will show that if  $\gamma(t) \in I^-(p, M)$  for some  $t$  then  $\gamma((-\infty, t]) \subseteq I^-(p, M)$ . This will mean that  $\gamma$  is totally past imprisoned in the set  $\overline{I^-(p, M)}$  which is compact since  $M$  is causally compact. To this end, let  $\gamma: (a, t] \rightarrow \Xi(M)$  be the maximal past geodesic extension of a generator of  $H^+(S_1)$ , with  $\gamma(t) \in I^-(p)$ . By [28, Proposition 14.53],  $H^+(S_1)$  and  $S_1$  are disjoint, so  $\gamma$  does not intersect  $S_1$ . Moreover, when followed to the past  $\gamma$  cannot intersect  $S_2$  since it would need to do so with the wrong time orientation. Hence  $\gamma$  stays in  $M$ . Let  $s < t$ . Then there is a causal curve in  $M$  from  $\gamma(s)$  to  $p$  formed by concatenating  $\gamma$  with a timelike curve from  $\gamma(t)$  to  $p$ . Such a curve exists since we assumed that  $\gamma(t) \in I^-(p, M)$ . Since this curve is not everywhere lightlike, there is a timelike curve from  $\gamma(s)$  to  $p$ . Hence  $\gamma(s) \in I^-(p)$ , proving the claim.

**Claim III:  $\mathcal{H} := H^+(S_1) \cap I^-(p, M)$  is past null geodesically ruled**

Let  $\mathcal{H}$  denote the set  $H^+(S_1) \cap I^-(p, M)$ . To find a past complete null geodesic segment through a point  $q \in \mathcal{H}$ , consider the intersection of  $\mathcal{H}$  and the generator of  $H^+(S_1)$  through  $q$ . This curve is a geodesic segment, and it is connected and past complete by the previous claim. Hence  $\mathcal{H}$  is past null geodesically ruled.

**Claim IV: The existence of  $\mathcal{H}$  is contradictory**

Since  $I^-(p, M)$  is open, the set  $\mathcal{H}$  is an open subset of  $H^+(S_1)$ . Since  $M$  is causally compact, the set  $\overline{I^-(p, M)}$  is compact. Hence  $\mathcal{H}$  is contained in a compact set and so has compact closure. Theorem 1.3.41 tells us that  $\mathcal{H}$  is a totally geodesic smooth null hypersurface. The inequality (1.3) from Section 1.3.1.1 then reads

$$\text{Ric}(K, K) \leq 0$$

for all null tangent vectors  $K$  to  $\mathcal{H}$ . Combining this with the lightlike convergence condition, we can conclude that

$$\text{Ric}(K, K) = 0.$$

We will now derive a contradiction from this.

If the spacetime were to satisfy the strict lightlike convergence condition, then the contradiction is immediate. When we assume that the lightlike generic condition holds, a further argument is needed. By Lemma 1.3.39 there is a dense, and in particular nonempty, subset of  $\mathcal{H}$  consisting of points on maximal null geodesics which are contained in  $\mathcal{H}$ . Choose one such maximal geodesic  $\gamma$ . Let  $b$  denote the null Weingarten map with respect to a null vector field  $K$  which agrees with the tangent vector field of  $\gamma$  with an affine parametrization. Since  $\mathcal{H}$  is totally geodesic,  $b = 0$  along  $\gamma$ . Equation (1.1) then implies that  $R(X, K)K$  is parallel to the null vector  $K$  for every vector  $X$ , where  $R$  denotes the curvature tensor of the spacetime. Given a vector  $X$ , denote by  $\lambda_X$  the real number such that  $R(X, K)K = \lambda_X K$ . Let  $k = g(K, \cdot)$  denote the covector dual to  $K$ . Consider the tensor  $G = k(\cdot)g(R(\cdot, K)K, \cdot)k(\cdot)$ . Evaluating  $G$  on  $(X, Y, Z, W)$  yields

$$k(X)g(R(Y, K)K, Z)k(W) = k(X)g(\lambda_Y K, Z)k(W) = k(X)\lambda_Y k(Z)k(W).$$

From this we see that the antisymmetrization of  $G(X, Y, Z, W)$  in  $Z$  and  $W$  is zero. Further antisymmetrizing in  $X$  and  $Y$  then also yields zero. In coordinates, this tells us that

$$K^e K^f K_{[a} R_{b]ef[c} K_{d]} = 0$$

on  $\gamma$ . However, the lightlike generic condition says that this antisymmetrized tensor is nonzero at some point along each maximal lightlike geodesic. We have now obtained a contradiction, so the assumption that  $H^+(S_1)$  is nonempty must be false. Hence  $M$  is globally hyperbolic.  $\square$

The above theorem is in one sense the strongest result one may hope for: The null energy condition is the weakest of the commonly used energy conditions, and in the setting of cobordisms it implies global hyperbolicity, which is the strongest of the commonly used causality conditions.

What we have proved is that topologically nontrivial Lorentzian cobordism cannot satisfy certain energy conditions. This is in sharp contrast to the result by Nardmann [26, Theorem 8.4] which allows the construction of Lorentzian metrics satisfying even stronger energy conditions on a large class of topologically nontrivial cobordisms. The difference is that the metrics so constructed only make the spacetime into a weak Lorentzian cobordism. This means that the notions of Lorentzian cobordism and weak Lorentzian cobordism are very different when one considers energy conditions. For a further discussion of this point, see [26, Section 9]. Note that a weak Lorentzian cobordism may be transformed into a Lorentzian cobordism by a homotopy of the metric making the light cones narrower. The theorems of Tipler (Theorem 1.3.46) and Nardmann ([26, Theorem

8.4]) then show that the imposed energy conditions must cease to hold at some point during this homotopy.

## Generic metrics satisfy the generic condition

In singularity theorems (for instance Theorem 2, Chapter 8, p. 266 of [19]) a condition called the *generic condition* is imposed on the spacetime. This is the condition that for each timelike or null geodesic  $\gamma$  there is some point  $\gamma(t)$  at which  $\dot{\gamma}^e \dot{\gamma}^f \dot{\gamma}_{[a} R_{b]ef[c} \dot{\gamma}_{d]} \neq 0$ . (See Definition A.3.5.) We follow the terminology of [1] in saying that a vector  $K$  for which  $K^e K^f K_{[a} R_{b]ef[c} K_{d]} \neq 0$  is *generic*.

It has been proposed (see for instance [19] p. 101 and [4] p. 21) that this "generic condition" should in some sense be "generic" among Lorentzian metrics on a given manifold. As pointed out by Lerner [23], the condition is generic in the set of metrics which also satisfies the strong energy condition. We will show that it is generic in the class of all metrics. The result we are going to prove is Theorem 2.6.3, which states the following.

**Theorem 2.6.3.** *Let  $M$  be a smooth manifold of dimension  $n \geq 4$ , possibly with boundary. Let  $L$  denote the bundle of Lorentzian metrics on  $M$  and endow the set  $\Gamma^\infty(L)$  of smooth sections with the Whitney  $C^r$  topology for any  $4 \leq r \leq \infty$ . Then there is a residual (and hence dense) set  $G \subseteq \Gamma^\infty(L)$  such that if  $g \in G$  and  $\gamma$  is a geodesic then the points where  $\dot{\gamma}^e \dot{\gamma}^f \dot{\gamma}_{[a} R_{b]ef[c} \dot{\gamma}_{d]}$  is nonzero along  $\gamma$  form a dense set. In particular,*

- *the lightlike generic condition holds for each metric  $g \in G$ ,*
- *the timelike generic condition holds for each metric  $g \in G$ ,*
- *the spacelike generic condition holds for each metric  $g \in G$ .*

An outline of the proof is provided in Section 2.1. The remaining sections contain a more detailed version of the proof. The method of proof is an adaptation of the method used by Rendall in [30].

### 2.1 Outline of the argument

In this section, we will outline the idea of the proof of Theorem 2.6.3, ignoring several technical complications.

Given a curvature tensor  $R$ , define  $F: TM \rightarrow T^4(TM)$  by

$$F(K)_{abcd} := K^e K^f K_{[a} R_{b]ef[c} K_{d]}.$$

We wish to show that the complement of the zero set of the function  $t \mapsto F(\dot{\gamma}(t))$  is dense for each geodesic  $\gamma$ . This means that it will be bad to have a large set of vectors for which the function vanishes to both zeroth and first order, i.e. vectors  $X$  which are zeros of  $F$  and for which the derivative of  $t \mapsto F(\dot{\gamma}(t))$  along the geodesic  $\gamma$  with initial velocity  $X$  is also zero. For if  $X$  does not belong to this set, then either  $X$  or some vector close to  $X$  on the same geodesic is generic. Our task is then to find metrics where this set of bad vectors is as small as possible, preferably empty.

It is fruitful to turn the question around. Given a vector  $X$ , which metrics are the ones for which  $X$  is a bad vector? (We outline here the argument only for non-lightlike vectors; the case of lightlike vectors is marginally more complicated.) By a computation by Beem and Harris [1] the condition that a non-lightlike vector  $X_p$  is nongeneric for some certain metric  $g$  is equivalent to that the curvature tensor  $R$  of  $g$  satisfies that

$$\langle \widehat{R}(A_p \wedge X_p), B_p \wedge X_p \rangle = 0 \quad \text{for all vectors } A_p \text{ and } B_p.$$

Here  $\widehat{R}$  is the tensor in  $S^2(\wedge^2(TM))$  given by the curvature tensor. (See Section 2.3 for details.) The condition that the derivative of  $F$  along the geodesic with initial velocity  $X$  is zero reduces to

$$\langle \widehat{\nabla_X R}(A_p \wedge X_p), B_p \wedge X_p \rangle = 0 \quad \text{for all vectors } A_p \text{ and } B_p.$$

Again,  $\widehat{\nabla_X R}$  is a tensor constructed from  $\nabla_X R$  as described in Section 2.3. The first equation determines a submanifold of codimension  $n(n-1)/2$  of the space  $\mathcal{R}^0$  of tensors with some of the symmetries of curvature tensors. The second equation determines a submanifold of codimension  $n(n-1)/2$  of the space  $\mathcal{R}^1$  of tensors with some of the symmetries of covariant derivatives of curvature tensors. This means that the set of pairs  $(R, \nabla R)$  induced by metrics which are bad for  $X$  is a codimension  $n(n-1)$  submanifold of  $\mathcal{R}^0 \oplus \mathcal{R}^1$ .

The previous paragraph considered a single vector. What we want is an argument which takes care of all vectors at once. We want to consider the set of pairs  $(g, X)$  where  $X$  is a bad vector for  $g$ . As an intermediate step, we consider the subset of  $TM \oplus \mathcal{R}^0 \oplus \mathcal{R}^1$  consisting of triples  $(X, R, \nabla R)$  satisfying the above equations, i.e. such that  $X$  is bad for the curvature tensor  $R$  with derivative  $\nabla R$ . This set is also a submanifold of codimension  $n(n-1)$ . Denote this submanifold  $\mathcal{NG}$  (for "non-generic"). We wish to use this set to determine which the bad pairs  $(g, X)$  are. To determine if  $X$  is bad for  $g$ , what is needed is enough information about  $g$  to compute the value of the curvature tensor and its first covariant derivative



at the basepoint of  $X$ . In other words, the derivatives of  $g$  up to third order are needed. This information is encoded in the bundle  $J^3L$  of 3-jets of Lorentzian metrics. What we want to find is then the subset of  $J^3L \oplus TM$  consisting of pairs  $(j^3g, X)$  such that  $(X, R, \nabla R) \in \mathcal{NG}$ , where  $R$  and  $\nabla R$  are the pointwise curvature tensor and its derivative computed from  $g$  at the basepoint of  $X$ . To this end, let  $R^0$  be the map computing  $R$  from  $j^3g$ , and  $R^1$  the map computing  $\nabla R$  from  $j^3g$ . Then  $W := (\text{id}_{TM}, R^0, R^1)^{-1}(\mathcal{NG})$  is the set of bad pairs  $(j^3g, X)$ . It turns out that the map  $(\text{id}_{TM}, R^0, R^1)$  is a submersion, so  $W$  is a submanifold of the same codimension as  $\mathcal{NG}$ , i.e. of codimension  $n(n-1)$ .

Now  $(j_x^3g, X) \in W$  means that  $(g, X)$  is a bad pair, so if we can arrange so that  $N \cap W = \emptyset$  where

$$N = \{(j_x^3g, X) \in J^3L \oplus TM \mid x \in M, X \in T_xM \setminus \{0\}\},$$

then there would be no bad vectors for  $g$  at all. Now  $N$  has dimension  $2n$  and  $W$  has codimension  $n(n-1)$ . If  $n(n-1) > 2n$ , i.e. if  $n > 3$ , then saying that the intersection  $N \cap W$  is empty is the same thing as saying that it is transverse. More specifically, it is the same thing as saying that the map  $\{X_x \in TM \mid X_x \neq 0\} \rightarrow J^3L \oplus TM$  defined by  $X_x \mapsto (j_x^3g, X_x)$  is transverse to  $W$ .

Finally, an application of a transversality theorem tells us that the set of such metrics is dense in the set  $\Gamma^\infty(L)$  of all metrics, completing the proof. See Section 2.6 for details.

In the outline just given, we have ignored many technical complications which are treated in more detail in the following sections.

## 2.2 Construction of the bundles $\mathcal{R}^i$ and $\mathcal{B}^i$

Given a real inner product space  $V$ , let  $\wedge^2 V$  denote the vector space of skew-symmetric bivectors constructed from  $V$  with the inner product

$$\langle A \wedge B, C \wedge D \rangle = \langle A, C \rangle \langle B, D \rangle - \langle A, D \rangle \langle B, C \rangle.$$

Let  $S^2(\wedge^2 V)$  denote the set of symmetric 2-tensors on  $\wedge^2 V$ . Clearly,  $S^2(\wedge^2 V)$  is a subspace of the space of  $(2, 0)$  tensors on  $\wedge^2 V$ .

Let  $M$  be a manifold. We will define four tensor bundles over  $M$ . The first two are  $\mathcal{R}^0$  and  $\mathcal{R}^1$ . We will think of  $\mathcal{R}^0$  as a bundle of tensors with some of the symmetries of Riemannian curvature tensors  $R$ . Similarly, we will think of  $\mathcal{R}^1$  as a bundle of tensors with some of the symmetries of the covariant derivative  $\nabla R$  of a curvature tensor. Explicitly, we use the following definition.

**Definition 2.2.1.** Let  $M$  be a smooth manifold. Then  $\mathcal{R}^0$  and  $\mathcal{R}^1$  are defined by

$$\begin{aligned}\mathcal{R}^0 &= S^2\left(\bigwedge^2 TM\right), \\ \mathcal{R}^1 &= S^2\left(\bigwedge^2 TM\right) \otimes T^*M.\end{aligned}$$

We will now construct bundles  $\mathcal{B}^0$  and  $\mathcal{B}^1$  with the help of two maps  $b^0$  and  $b^1$ .

**Definition 2.2.2.** Define a map  $b^0: \mathcal{R}^0 \rightarrow \bigwedge^4 TM$  (in each fiber) by antisymmetrization over the last three components:

$$b^0(R)_{abcd} = R_{a[bcd]}.$$

Similarly, define a map  $b^1: \mathcal{R}^1 \rightarrow (\bigwedge^4 TM \otimes T^*M) \oplus T^5(TM)$  in each fiber by

$$b^1(R)_{abcde} = (R_{a[bcd]e}, R_{ab[cde]}).$$

Let

$$\mathcal{B}^0 = (b^0)^{-1}(\{0\})$$

and

$$\mathcal{B}^1 = (b^1)^{-1}(\{0\}).$$

**Remark 2.2.3.** The idea is that the zero set  $\mathcal{B}^0$  of  $b^0$  consists of those tensors which satisfy the first Bianchi identity. Similarly, the zero set  $\mathcal{B}^1$  of  $b^1$  is the set of tensors which satisfy both the first and second Bianchi identities.

**Remark 2.2.4.** The maps  $b^0$  and  $b^1$  are vector bundle maps of constant rank, so  $\mathcal{B}^0$  and  $\mathcal{B}^1$  are vector bundles over  $M$ .

**Remark 2.2.5.** We will denote elements of  $\mathcal{B}^1$  by  $\nabla S$ . This notation is purely formal; the  $\nabla$  does not signify an actual covariant derivative. The purpose of the notation is to remind us that  $\nabla S$  is a tensor with the same symmetries as a covariant derivative of a curvature tensor. Given a vector  $X$ , we will let  $\nabla_X S$  denote the tensor  $S_{abcde}X^e$ , where  $S_{abcde}$  are the components of  $\nabla S$ . Note that  $\nabla_X S \in \mathcal{B}^0$ .

## 2.3 Viewing a curvature tensor $R$ as an endomorphism $\hat{R}$

In this section, we will describe a way of interpreting a tensor  $R \in \mathcal{B}^0$  as an endomorphism of  $\bigwedge^2(TM)$ . The isomorphism used for this will depend on a choice of metric. Moreover, given a vector  $X \in T_p M$  and a tensor  $\nabla S \in \mathcal{B}^1$ , the tensor  $\nabla_X S$  is an element of  $\mathcal{B}^0$  and as such may be viewed as an endomorphism of  $\bigwedge^2(TM)$  in the same way.

**Definition 2.3.1.** Given a metric  $g$ , define a map  $\widehat{g}: \mathcal{B}^0 \rightarrow \text{End}(\wedge^2(TM))$  by letting  $\widehat{g}(R)$  be the endomorphism such that

$$\langle \widehat{g}(R)(A \wedge B), C \wedge D \rangle = R(A, B, C, D) \quad \text{for all } A, B, C, D \in T_p M \text{ and all } p \in M.$$

When the metric is implicitly understood, we will use the notation  $\widehat{R} := \widehat{g}(R)$ . For  $\nabla S \in \mathcal{B}^1$  and given a vector field  $X$ , we will use the notation  $\widehat{\nabla_X S} = \widehat{g}(\nabla_X S)$ .

## 2.4 Construction of $\mathcal{NG}^{i,t}$ and $\mathcal{NG}^{i,\ell}$

### 2.4.1 Definitions

Let  $\mathbb{P}TM$  denote the projectivization of the tangent bundle  $TM$  of  $M$ . We will now construct certain subspaces of  $L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$ . We will denote an element of  $L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  by  $(g_p, [X], R, \nabla S)$ , i.e. including the basepoint  $p$  only in the notation for  $g_p$ . We emphasize again that the notation  $\nabla S$  is purely formal.

The following are the spaces we are interested in:

**Definition 2.4.1.** Let  $\mathcal{NG}^{0,t} \subseteq L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  be the set of  $(g_p, [X], R, \nabla S)$  such that

- $g_p(X, X) < 0$ ,
- $\langle \widehat{R}(A \wedge X), B \wedge X \rangle = 0$  for all  $A, B \in T_p M$ .

**Definition 2.4.2.** Let  $\mathcal{NG}^{1,t} \subseteq L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  be the set of  $(g_p, [X], R, \nabla S)$  such that

- $g_p(X, X) < 0$ ,
- $\langle \widehat{(\nabla_X S)}(A \wedge X), B \wedge X \rangle = 0$  for all  $A, B \in T_p M$ .

**Definition 2.4.3.** Let  $\mathcal{NG}^{0,\ell} \subseteq L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  be the set of  $(g_p, [X], R, \nabla S)$  such that

- $g_p(X, X) = 0$ ,
- $\langle \widehat{R}(A \wedge X), B \wedge X \rangle = 0$  for all  $A, B \in T_p M$  orthogonal to  $X$  (with respect to  $g_p$ ).

**Definition 2.4.4.** Let  $\mathcal{NG}^{1,\ell} \subseteq L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  be the set of  $(g_p, [X], R, \nabla S)$  such that

- $g_p(X, X) = 0$ ,
- $\langle \widehat{(\nabla_X S)}(A \wedge X), B \wedge X \rangle = 0$  for all  $A, B \in T_p M$  orthogonal to  $X$  (with respect to  $g_p$ ).

Some words about these definitions are in order. First note that the conditions  $g_p(X, X) < 0$  and  $g_p(X, X) = 0$  are independent of the choice of representative  $X$  of  $[X] \in \mathbb{P}TM$ . Similarly, the conditions of the forms  $\langle \widehat{R}(A \wedge X), B \wedge X \rangle = 0$  and  $\langle \widehat{\nabla_X S}(A \wedge X), B \wedge X \rangle = 0$  are also independent of the choice of representative. Note also that an element of  $L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  specifies a metric so that the construction of  $\widehat{R}$  and  $\widehat{\nabla_X S}$  as in Section 2.3 is well-defined.

### 2.4.2 Manifoldness and codimension

We now wish to show that the spaces defined above are submanifolds, and compute their codimensions.

**Lemma 2.4.5.**  $\mathcal{NG}^{0,t}$  is a submanifold of  $L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  of codimension  $n(n-1)/2$ .

*Proof.* It is sufficient to show this locally, in a local trivialization of the bundle  $L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$ . We will choose a suitable such trivialization shortly.

First let

$$\mathcal{TLV} = \{(g, [X]) \in L \oplus \mathbb{P}TM \mid [X] \text{ is timelike for } g\}.$$

This is a subbundle of  $L \oplus \mathbb{P}TM$ , of codimension 0, so it is sufficient to show that  $\mathcal{NG}^{0,t}$  is a submanifold of  $\mathcal{TLV} \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  of the desired codimension  $n(n-1)/2$ . Note that by definition,  $\mathcal{NG}^{0,t} \subseteq \mathcal{TLV} \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$ .

**Proof part I(Choice of trivialization):** Choose an arbitrary point  $p \in M$ , and let  $U$  be an open neighborhood of  $p$  such that the bundles  $\mathcal{TLV}, \mathcal{B}^0, \mathcal{B}^1$  and  $TM$  are all trivializable over  $U$ . Let  $\Psi: U \times \mathcal{TLV}_p \rightarrow \mathcal{TLV}$  be an arbitrary local trivialization over  $U$  of the bundle  $\mathcal{TLV}$ . We wish to extend this to a local trivialization of the full bundle in a clever way. Since  $\mathcal{B}^0$  and  $\mathcal{B}^1$  are vector subbundles of tensor bundles on  $TM$ , extending  $\Psi$  to a trivialization of  $\mathcal{TLV} \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  may be done by, for each  $(g, [X]) \in \mathcal{TLV}_p$ , choosing a frame  $V^{g, [X]}$  on  $U$ , and expressing each fiber in the basis supplied by this frame. The frame  $V^{g, [X]}$  must depend smoothly on  $(g, [X])$  for the extended trivialization to be smooth. We propose to choose  $V^{g, [X]} = \{X_1, X_2, \dots, X_n\}$  in the following manner:

- At the point  $p$ , choose representatives  $X_n \in [X]$  for each  $[X]$ , depending smoothly on  $[X]$ .
- At the point  $p$ , choose (smoothly in  $[X]$ )  $n-1$  additional vectors  $Y_1, \dots, Y_{n-1}$  which together with  $X_n$  form a basis.
- Extend these vectors to frames, by use of a local trivialization of  $TM$  over  $U$ .

- Perform the Gram-Schmidt orthogonalization procedure to obtain the required orthogonality relations. This algorithm is smooth in the components of the metric  $g$  and in the starting vector  $X_n$ .

Using this frame, we can form the ordered frame for  $\wedge^2 TM$  given by

$$X_1 \wedge X_2, X_1 \wedge X_3, X_2 \wedge X_3; \dots; X_1 \wedge X_n, X_2 \wedge X_n, \dots, X_{n-1} \wedge X_n.$$

For our purposes, the frame needs to be ordered so that the last  $n-1$  basis vectors are  $X_i \wedge X_n$  for  $1 \leq i < n$ .

**Proof part II(Concrete description of  $\mathcal{NG}^{0,t}$  in the chosen frame):** We now express the condition  $\langle \hat{R}(A \wedge X), B \wedge X \rangle = 0$  in the basis provided by our local trivialization, where we have fixed  $p \in U$  and  $(g_p, [X]) \in \mathcal{TLV}_p$ . Since  $\hat{R}$  is a self-adjoint linear operator on  $\wedge^2 TM$ , it can be represented by a symmetric matrix in the chosen basis. We write this on block form where the lower right-hand block has size  $(n-1) \times (n-1)$ :

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

The condition defining  $\mathcal{NG}^{0,t}$  (that  $\langle \hat{R}(A \wedge X), B \wedge X \rangle = 0$  for all  $A, B \in T_p M$ ) is equivalent to the condition that  $\langle \hat{R}(X_i \wedge X), X_j \wedge X \rangle = 0$  for all  $i$  and  $j$ . We ordered the basis of  $\wedge^2 TM$  so that the vectors  $X_i \wedge X$  are the last  $n-1$  vectors in the basis. Hence the condition is equivalent to  $C = 0$ , when  $\hat{R}$  is expressed in this basis.

**Proof part III(Definition of a map  $c$ ):** For each  $0 \leq i \leq j < n$ , we define a map  $c_{i,j}: \mathcal{TLV} \oplus \mathcal{B}^0 \oplus \mathcal{B}^1|_U \rightarrow \mathbb{R}$  by letting  $c_{i,j}(g_p, [X], R, \nabla S)$  be the  $(i, j)$  component of the part  $C$  of the matrix representation of  $\hat{R}$  at the point  $p$  using the basis from the fixed frame at  $p$ . In other words, let

$$c_{i,j}(g_p, [X], R, \nabla S) = \langle \hat{R}(X_i \wedge X_n), X_j \wedge X_n \rangle \quad \text{for } 0 \leq i \leq j < n.$$

Note that we have fixed a frame  $V^{g,[X]}$  so that these functions  $c_{ij}$  are well-defined. In particular, they depend up to a multiplicative scaling on the representative  $X_n \in [X]$  chosen, and we have already fixed such a representative. Define the map  $c: \mathcal{TLV} \oplus \mathcal{B}^0 \oplus \mathcal{B}^1|_U \rightarrow \mathbb{R}^{n(n-1)/2}$  by  $c = (c_{1,1}, c_{1,2}, \dots, c_{n-1,n-1})$ . Clearly,  $\mathcal{NG}^{0,t}|_U$  is  $c^{-1}(0)$ . If we can show that  $c$  is a submersion, then we know that  $\mathcal{NG}^{0,t}|_U$  is a codimension  $n(n-1)/2$  submanifold of  $\mathcal{TLV} \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$ , since  $\{0\}$  is a codimension  $n(n-1)/2$  submanifold of  $\mathbb{R}^{n(n-1)/2}$ .

**Proof part IV(The map  $c$  is a submersion):** For  $c$  to be a submersion at some point  $(g_p, [X]_p, R_p, \nabla S_p)$ , it is sufficient that the restricted map  $c|_{\{g_p\} \times \{[X]_p\} \times B_p^0 \times \{\nabla S_p\}}$  is a submersion. This map is linear, so it is enough to show that it is surjective. The

relations defining  $\mathcal{B}^0$  as a subspace of all 2-tensors translate (via  $\langle \widehat{R}(A \wedge B), C \wedge D \rangle = \langle R(A, B)C, D \rangle$ ) to

$$\begin{aligned}\widehat{R}(A \wedge B) &= -\widehat{R}(B \wedge A), \\ \langle \widehat{R}(A \wedge B), C \wedge D \rangle &= \langle \widehat{R}(C \wedge D), A \wedge B \rangle, \\ \langle \widehat{R}(A \wedge B), C \wedge D \rangle + \langle \widehat{R}(A \wedge C), D \wedge B \rangle + \langle \widehat{R}(A \wedge D), B \wedge C \rangle &= 0.\end{aligned}$$

We are interested in the ways in which these relations impose relations among the  $c_{i,j}$ ; in fact our goal is to show that no such relations exist. The first relation follows immediately from  $\widehat{R}$  being a tensor on  $\wedge^2 TM$ , and so gives no restriction on the  $c_{i,j}$ . The second tells us that the matrix representation of  $\widehat{R}$  is symmetric, but since we required that  $i \leq j$ , this puts no restriction on the  $c_{i,j}$ . The third condition comes from the Bianchi identity, i.e. the definition of  $\mathcal{B}^0$  as the kernel of the map  $b^0$ . The components  $c_{i,j}$  all refer to entries corresponding to basis bivectors  $X_k \wedge X_n$  for  $k < n$ . Explicitly

$$c_{i,j} = \langle \widehat{R}(X_j \wedge X_n), X_i \wedge X_n \rangle.$$

For such vectors

$$\langle \widehat{R}(X_j \wedge X_n), X_i \wedge X_n \rangle + \langle \widehat{R}(X_j \wedge X_i), X_n \wedge X_n \rangle + \langle \widehat{R}(X_j \wedge X_n), X_n \wedge X_i \rangle = 0,$$

so the third condition gives no relations between the  $c_{i,j}$ . This means that given any choice of components  $c_{i,j}$  (with  $1 \leq i \leq j < n$ ) there is an element of  $\mathcal{B}^0$  having these components in its lower right block. In other words, the linear map  $c|_{\{g_p\} \times \{[X]_p\} \times \mathcal{B}_p^0 \times \{\nabla S_p\}}$  is surjective and hence a submersion, so the full map  $c$  is also a submersion.

As noted above,  $\mathcal{N}\mathcal{G}^{0,t} = c^{-1}(\{0\})$ . Since  $c$  is a submersion with image  $\mathbb{R}^{n(n-1)/2}$ , this means that  $\mathcal{N}\mathcal{G}^{0,t}$  is a submanifold of  $\mathcal{TLV} \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  of codimension  $n(n-1)/2$ , and hence also a submanifold of  $L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  of codimension  $n(n-1)/2$ .  $\square$

**Lemma 2.4.6.** *The set  $\mathcal{N}\mathcal{G}^{1,t}$  is a submanifold of  $L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  of codimension  $n(n-1)/2$ .*

*Proof.* The proof of this lemma is similar to that of Lemma 2.4.5. Choose the family of frames  $V^{g,[X]}$  in the same way. With this choice,  $\nabla_{X_n} S \in \mathcal{B}^1$  may be expressed as a block matrix, the same way  $R \in \mathcal{B}^0$  way expressed as a block matrix in the proof of Lemma 2.4.5. Just as in the preceding proof, define the functions  $c_{i,j}$  to be the components of the lower right-hand block. We need the map  $c = (c_{1,1}, c_{1,2}, \dots, c_{n-1,n-1}): \mathcal{TLV} \oplus \mathcal{B}^0 \oplus \mathcal{B}^1 \rightarrow \mathbb{R}^{n(n-1)/2}$  to be a submersion. The same argument as in the proof of Lemma 2.4.5 shows that the first Bianchi identity does not impose any relations between the  $c_{i,j}$ . Similarly, the second Bianchi identity only imposes relations between  $S_{abcde}$  for different  $e$ , never relations between

components with the same  $e$ . In particular, since the  $c_{ij}$  are defined using  $S_{abcdn}$ , no relations between them are imposed by the second Bianchi identity. Hence  $c$  is a submersion, and we conclude that  $\mathcal{NG}^{1,t}$  has codimension  $n(n-1)/2$ .  $\square$

**Proposition 2.4.7.**  *$\mathcal{NG}^{0,t} \cap \mathcal{NG}^{1,t}$  is a submanifold of  $L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  of codimension  $n(n-1)$ .*

*Proof.* Since  $\mathcal{NG}^{0,t}$  is of the form  $E_0 \oplus \mathcal{B}^1$  for some fiber bundle  $E_0 \rightarrow L \oplus \mathbb{P}TM$ , and  $\mathcal{NG}^{1,t}$  is of the form  $E_1 \oplus \mathcal{B}^0$  for some fiber bundle  $E_1 \rightarrow L \oplus \mathbb{P}TM$ , their intersection  $\mathcal{NG}^t = \mathcal{NG}^{0,t} \cap \mathcal{NG}^{1,t}$  is the product bundle  $E_0 \oplus E_1$ , which is a submanifold of  $L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  of codimension  $\text{codim } \mathcal{NG}^{0,t} + \text{codim } \mathcal{NG}^{1,t} = n(n-1)$ .  $\square$

To compute the codimensions of  $\mathcal{NG}^{0,\ell}$  and  $\mathcal{NG}^{1,\ell}$  we proceed similarly. However, we may obtain a sharper bounds on the codimensions by making use of the fact that these spaces are defined in terms of lightlike vectors. Let  $\mathcal{LLV}$  denote the bundle

$$\mathcal{LLV} = \{(g, [X]) \in L \oplus \mathbb{P}TM \mid [X] \text{ is lightlike for } g\}.$$

Then  $\mathcal{LLV}$  has codimension 1 in  $L \oplus \mathbb{P}TM$ .

**Lemma 2.4.8.** *The set  $\mathcal{NG}^{0,\ell}$  is a submanifold of  $\mathcal{LLV} \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  of codimension  $(n-1)(n-2)/2$ .*

*Proof.* The proof is analogous to that of Lemma 2.4.5, with the exception that the frame  $V^{g,[X]}$  is chosen so that  $X_2, \dots, X_{n-1}$  are all orthogonal to  $X_n$ , but  $X_1$  need not be. This means that when  $\hat{R}$  is written in block form

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

where lower right-hand block  $C$  is a  $(n-2) \times (n-2)$  matrix, the equation describing  $\mathcal{NG}^{0,\ell}$  is precisely that  $C$  is zero. As in the proof of Lemma 2.4.5, define functions  $c_{i,j}$  by extracting the  $(i, j)$  component of  $C$ . This time, since  $C$  is smaller, we get  $(n-1)(n-2)/2$  different functions  $c_{i,j}$  (after considering the symmetry of  $\hat{R}$ ). As before, these are independent in the sense that  $c = (c_{1,1}, c_{1,2}, \dots, c_{n-1,n-1})$  is a submersion. Hence  $\mathcal{NG}^{0,\ell}$  has codimension  $(n-1)(n-2)/2$  in  $\mathcal{LLV} \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$ .  $\square$

**Lemma 2.4.9.** *The set  $\mathcal{NG}^{1,\ell}$  is a submanifold of  $\mathcal{LLV} \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  of codimension  $(n-1)(n-2)/2$ .*

*Proof.* To compute the codimension of  $\mathcal{NG}^{1,\ell}$ , do the same thing as for  $\mathcal{NG}^{1,t}$ , but in the bundle  $\mathcal{LLV} \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  and with a block  $C$  of size  $(n-2) \times (n-2)$ . This yields a codimension of  $(n-1)(n-2)/2$ .  $\square$

**Proposition 2.4.10.**  $\mathcal{NG}^{0,\ell} \cap \mathcal{NG}^{1,\ell}$  is a submanifold of  $L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  of codimension  $(n-1)(n-2) + 1$ .

*Proof.* Since  $\mathcal{NG}^{0,\ell}$  and  $\mathcal{NG}^{1,\ell}$  impose restrictions in different components of the product  $\mathcal{LLV} \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$ , their intersection  $\mathcal{NG}^\ell = \mathcal{NG}^{0,\ell} \cap \mathcal{NG}^{1,\ell}$  is a submanifold of codimension  $\text{codim} \mathcal{NG}^{0,\ell} + \text{codim} \mathcal{NG}^{1,\ell} = (n-1)(n-2)$  in  $\mathcal{LLV} \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$ . Since  $\mathcal{LLV} \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  has codimension 1 in  $L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$ , we conclude that  $\mathcal{NG}^\ell$  has codimension  $(n-1)(n-2) - 1$  in  $L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$ .  $\square$

## 2.5 The map $\alpha$

The purpose of this section is to define a bundle map

$$\alpha: \mathbb{P}TM \oplus J^3L \rightarrow \mathbb{P}TM \oplus L \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$$

and show that it is a submersion. Recall that the jet bundle  $J^3L$  has canonical projections  $\pi_i: J^3L \rightarrow J^iL$  for  $i = 0, 1, 2$ . These are described in Section A.5.

**Definition 2.5.1.** Let  $\alpha$  be the bundle map

$$\mathbb{P}TM \oplus J^3L \xrightarrow{(\text{id}_{\mathbb{P}TM}, \pi_0, R^0, R^1)} \mathbb{P}TM \oplus L \oplus \mathcal{B}^0 \oplus \mathcal{B}^1.$$

Here  $J^3L$  denotes the bundle of 3-jets of metrics on  $M$ , and  $\mathbb{P}TM$  denotes the projectivized tangent bundle of  $M$ . The map  $\text{id}_{\mathbb{P}TM}$  is the identity on  $\mathbb{P}TM$ . The map  $\pi_0: J^3L \rightarrow L$  is the projection onto the 0-jet part of  $J^3L$ . (Note that we use the canonical isomorphism between  $L$  and  $J^0L$  to view  $\pi_0$  as a map  $J^3L \rightarrow L$ .) The map  $R^0: J^3L \rightarrow \mathcal{B}^0$  computes the Riemann curvature tensor of a metric. More specifically, if  $g \in J_p^3L$  then  $R^0(g)$  is the curvature tensor at  $p$  of any metric corresponding to the equivalence class  $g$ . Similarly, the map  $R^1: J^3L \rightarrow \mathcal{B}^1$  computes the first covariant derivative  $R^1(g) = \nabla R$  of the curvature tensor  $R$  of  $g$ . The 3-jet of a metric is precisely the information needed to compute these tensors.

We wish to show that  $\alpha := (\text{id}_{\mathbb{P}TM}, \pi_0, R^0, R^1)$  is a submersion. Since the map  $(\text{id}_{\mathbb{P}TM}, \pi_0, R^0, R^1)$  is simply the product of the map  $(\pi_0, R^0, R^1): J^3L \rightarrow L \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  with the identity  $\text{id}_{\mathbb{P}TM}: \mathbb{P}TM \rightarrow \mathbb{P}TM$ , and the identity is a submersion, it is sufficient to show that  $(\pi_0, R^0, R^1)$  is a submersion.

To show that  $(\pi_0, R^0, R^1)$  is a submersion, we will use the following lemma.

**Lemma 2.5.2.** Let  $B$ ,  $F$  and  $F'$  be smooth manifolds. Let  $E \xrightarrow{\pi} B$  and  $E' \xrightarrow{\pi'} B$  be smooth fiber bundles with fibers  $F$  and  $F'$ . Let  $\varphi: E \rightarrow E'$  be a smooth bundle map. Suppose that for each  $p \in B$  the fiberwise restriction  $\varphi: E_p \rightarrow E'_p$  is a submersion. Then  $\varphi$  is itself a submersion.



*Proof.* Choose some point  $e \in E$ , and let  $b = \pi(e)$ . Choose an open neighborhood  $U \subseteq B$  of  $b$  such that  $E$  and  $E'$  both have trivializations over  $U$ . Use this trivialization to express  $e = (b, f)$ . For each  $c \in U$ , the bundle map  $\varphi$  restricts to some map  $\varphi_c: \{c\} \times F \rightarrow \{c\} \times F'$ , i.e. to a map  $\varphi_c: F \rightarrow F'$ . By hypothesis, this map is a submersion, so there are coordinates on a neighborhood  $V_c \subseteq F$  of  $f$  and its image  $\varphi_c(V_c) \subseteq F'$  such that  $\varphi$  is the projection onto the first  $k$  coordinates (where  $k = \dim F - \dim F'$ ). We may choose these coordinate systems to depend smoothly on  $c$ , and (possibly after shrinking  $U$ ) such that  $V = \bigcap_{c \in U} V_c \neq \emptyset$ . In coordinates in the neighborhood  $U \times V$  of  $e$ , we then have

$$\varphi(b, f) = (b, \psi(f))$$

where  $\psi$  denotes projection onto the first  $k$  coordinates. Hence we can locally express  $\varphi$  as a product of the identity and a projection, both of which are submersions, so  $\varphi$  is a submersion.  $\square$

**Proposition 2.5.3.** *The map  $(\pi_0, R^0, R^1): J^3 L \rightarrow L \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  is a submersion.*

*Proof.* Note that  $J^3 L$  and  $L \oplus \mathcal{B}^0 \oplus \mathcal{B}^1$  are bundles over  $M$ , so the lemma tells us that  $(\pi_0, R^0, R^1)$  is a submersion if its restriction to the fiber over any point  $p \in M$  is a submersion. We then need to show that

$$(\pi_0, R^0, R^1): J_p^3 L \rightarrow L_p \oplus \mathcal{B}_p^0 \oplus \mathcal{B}_p^1$$

is a submersion. Note now that both  $J_p^3 L$  and  $L_p \oplus \mathcal{B}_p^0 \oplus \mathcal{B}_p^1$  are bundles over  $L_p$ . This means that the lemma tells us that it is sufficient to show, for each  $h \in L_p$ , that the restriction to the fiber over  $h$  is a submersion. Thus we have reduced the problem to showing that

$$(R^0, R^1): (J_p^3 L)_h \rightarrow \mathcal{B}_p^0 \oplus \mathcal{B}_p^1$$

is a submersion.

Choose some arbitrary point  $[g] \in (J_p^3 L)_h$ . We will show that  $(R^0, R^1)$  is a submersion at  $[g]$ . In fact, we will show that the restriction of  $(R^0, R^1)$  to the set  $J := \{[k] \in (J_p^3 L)_h: \pi_1(k) = \pi_1(g)\}$  is a submersion. The reason for this restriction is to make the 1-jets of the elements of the domain agree, so that we may choose normal coordinates: Choose coordinates centered  $p$  which are normal with respect to  $\pi_1(g)$ . In these coordinates, an element  $[g]$  of  $J$  is characterized by the tensors  $g_{ab,cd}$  and  $g_{ab,cde}$ . Similarly, an element of  $\mathcal{B}^0$  is a tensor  $R_{abcd}$  and an element of  $\mathcal{B}^1$  is a tensor  $R_{abcde}$ . (We use implicitly the isomorphisms mentioned in Section 2.3 to view  $\mathcal{B}^0$  and  $\mathcal{B}^1$  as spaces of 4- and 5-tensors, respectively. Since we have restricted our attention to the set  $J$ , these isomorphisms are canonical.)

In the center of normal coordinates it holds that the Riemann tensor of a metric  $g$  is

$$R_{abcd} = \Gamma_{abd,c} - \Gamma_{abc,d} = \frac{1}{2} (g_{ad,bc} + g_{bc,ad} - g_{ac,bd} - g_{bd,ac})$$

and that its first derivative is

$$R_{abcd;e} = R_{abcd,e} = \Gamma_{abd,ce} - \Gamma_{abc,de} = \frac{1}{2} (g_{ad,bce} + g_{bc,ade} - g_{ac,bde} - g_{bd,ace}).$$

These are then the coordinate expressions of  $R^0$  and  $R^1$  in normal coordinates at  $p$ . Note in particular that they are linear. This means that  $R^0$  and  $R^1$  agree with their differential maps at  $[g]$ . Note that the tangent space of  $(J_p^3 L)_h$  is the product  $T_Q^4 \times T_Q^5$  where  $T_Q^4$  is the vector space of 4-tensors with the symmetries of second derivatives of a metric, and  $T_Q^5$  is the vector space of 4-tensors with the symmetries of third derivatives of a metric. Explicitly, the tangent maps of  $R^0$  and  $R^1$  are then

$$R_*^0(Q_{abcd}) = \frac{1}{2} (Q_{adbc} + Q_{bcad} - Q_{acbd} - Q_{bdac}),$$

$$R_*^0(Q_{abcd,e}) = \frac{1}{2} (Q_{adbc,e} + Q_{bcad,e} - Q_{acbd,e} - Q_{bdac,e}).$$

It is now sufficient to show that  $(R_*^0, R_*^1)$  is a surjective linear map. The two maps are surjective (in fact isomorphisms) since the following maps are their inverses [27].

$$g_{ab,cd} = \frac{1}{3} (R_{acbd} + R_{adbc})$$

$$g_{ab,cde} = \frac{1}{6} \sum_{(ijk)} R_{aibj,k}$$

The sum ranges over all permutations  $ijk$  of  $cde$ . Note that these two maps are independent of each other: The map

$$(R_{\alpha\beta\gamma\delta}, R_{abcde}) \mapsto \left( \frac{1}{3} (R_{\alpha\gamma\beta\delta} + R_{\alpha\delta\beta\gamma}), \frac{1}{6} \sum_{(ijk)} R_{aibj,k} \right)$$

is an inverse of  $(R_*^0, R_*^1)$  and we have shown that it is a submersion. □

As argued earlier, this implies the following corollary.

**Corollary 2.5.4.** *The bundle map*

$$(\text{id}_{\mathbb{P}TM}, \pi_0, R^0, R^1): \mathbb{P}TM \oplus J^3 L \rightarrow \mathbb{P}TM \oplus L \oplus \mathcal{B}^0 \oplus \mathcal{B}^1.$$

*is a submersion.*

## 2.6 Transversality

An application of Proposition B.3.3 yields the following result.

**Proposition 2.6.1.** *Let  $M$  be a manifold without boundary. Let  $W \subseteq J^3L \oplus \mathbb{P}TM$  be a submanifold of codimension  $q$ . Then the set of metrics  $g \in \Gamma^\infty(L)$  such that the image of the map  $\mathbb{P}TM \rightarrow J^3L \oplus \mathbb{P}TM$  defined by  $V_x \mapsto (j^3g_x, V_x)$  is transverse to  $W$  is residual in the Whitney  $C^4$  topology on the space  $\Gamma^\infty(L)$  of all metrics.*

*Proof.* This is simply an application of Proposition B.3.3 with  $r = 3$  and  $K = \mathbb{P}TM$ .  $\square$

**Proposition 2.6.2.** *Let  $(m, g)$  be a Lorentzian manifold with curvature tensor  $R$ . Let  $\gamma$  be a geodesic with initial velocity  $X := \dot{\gamma}(0)$ . Let  $\rho(g): \mathbb{P}TM \rightarrow J^3L \oplus \mathbb{P}TM$  denote the map defined by  $\rho(g)(V_x) = (j^3g_x, V_x)$ . If either*

- *$X$  is timelike and  $\rho(g)([X]) \notin \alpha^{-1}(\mathcal{NG}^t)$ , or*
- *$X$  is lightlike and  $\rho(g)([X]) \notin \alpha^{-1}(\mathcal{NG}^\ell)$*

*then for every  $\delta > 0$  there is a point  $t \in (-\delta, \delta)$  such that  $\dot{\gamma}(t)$  is a generic vector.*

*Proof.* Consider the case when  $X$  is a lightlike vector. That  $\rho(g)([X]) \notin \alpha^{-1}(\mathcal{NG}^\ell)$  means that either  $\alpha(\rho(g)([X])) \notin \mathcal{NG}^{0,\ell}$  or  $\alpha(\rho(g)([X])) \notin \mathcal{NG}^{1,\ell}$ . In the first case, we know by definition of  $\alpha$  and  $\mathcal{NG}^{0,\ell}$  that there are vectors  $A$  and  $B$  orthogonal to  $X$  such that

$$\langle \widehat{R}(A \wedge X), B \wedge X \rangle \neq 0$$

where  $R$  is the curvature tensor corresponding to  $g$ . By Proposition 2.2 of [1], this means that  $X$  is a generic vector, and so we may choose  $t = 0$ .

The remaining case is when  $\alpha(\rho(g)([X])) \in \mathcal{NG}^{0,\ell}$  but  $\alpha(\rho(g)([X])) \notin \mathcal{NG}^{1,\ell}$ . By definition of  $\mathcal{NG}^{1,\ell}$  we then know that there are vectors  $E$  and  $F$  orthogonal to  $X$  such that

$$\langle \widehat{\nabla_X R}(E \wedge X), F \wedge X \rangle \neq 0.$$

By the same argument as is used in [1], this is equivalent to that there are vectors  $A, B, C$  and  $D$  such that

$$0 \neq X^e X^f X_{[a} (\nabla_X R)_{b]ef[c} X_{d]} A^a B^b C^c D^d.$$

Extend these four vectors by parallel transport to vector fields along  $\gamma$  in a neighborhood of  $\gamma(0)$ . Then  $\nabla_X A = \nabla_X B = \nabla_X C = \nabla_X D = 0$  so the above is equivalent to that

$$0 \neq X(k)$$

where  $k$  is the function along  $\gamma$  defined by

$$k = X^e X^f X_{[a} R_{b]e f [c} X_{d]} A^a B^b C^c D^d.$$

By definition, it holds that  $k(t) \neq 0$  only if  $\dot{\gamma}(t)$  is generic. Since we have now shown that  $\frac{dk}{dt}(0) = X(k) \neq 0$  and we assumed that  $k(0) = 0$ , there must be some point  $t$  in any neighborhood of 0 such that  $k(t) \neq 0$ . Hence  $\dot{\gamma}(t)$  is generic for arbitrarily small  $t$ .

The proof in the case when  $X$  is timelike is completely analogous. This completes the proof.  $\square$

We are now finally in a position to prove our main theorem.

**Theorem 2.6.3.** *Let  $M$  be a smooth manifold of dimension  $n \geq 4$ , possibly with boundary. Let  $L$  denote the bundle of Lorentzian metrics on  $M$  and endow the set  $\Gamma^\infty(L)$  of smooth sections with the Whitney  $C^r$  topology for any  $4 \leq r \leq \infty$ . Then there is a residual (and hence dense) set  $G \subseteq \Gamma^\infty(L)$  such that if  $g \in G$  and  $\gamma$  is a geodesic then the points where  $\dot{\gamma}^e \dot{\gamma}^f \dot{\gamma}_{[a} R_{b]e f [c} \dot{\gamma}_{d]}$  is nonzero along  $\gamma$  form a dense set. In particular,*

- the lightlike generic condition holds for each metric  $g \in G$ ,
- the timelike generic condition holds for each metric  $g \in G$ ,
- the spacelike generic condition holds for each metric  $g \in G$ .

*Proof.* We note that all of the constructions for the timelike case work equally well for the spacelike case, so we prove the result only for timelike and lightlike geodesics.

Consider the following diagram, where  $\alpha = (\text{id}_{\mathbb{P}TM}, \pi_0, R^0, R^1)$  is the map considered in Section 2.5.

$$\begin{array}{ccc} J^3 L \oplus \mathbb{P}TM & \xrightarrow{\alpha} & L \oplus \mathbb{P}TM \oplus \mathcal{B}^0 \oplus \mathcal{B}^1 \\ \uparrow & & \uparrow \\ \alpha^{-1}(\mathcal{NG}) & \xrightarrow{\alpha} & \mathcal{NG} \end{array}$$

Here  $\mathcal{NG}$  denotes the union  $\mathcal{NG}^t \cup \mathcal{NG}^\ell$ , so by the results of Section 2.4 we know that  $\mathcal{NG}$  is a union of submanifolds of codimensions (at least)  $n(n-1)$  and  $(n-1)(n-2)+1$ . Since  $\alpha$  is a submersion, as was shown in the previous section, and inverse submersions preserve codimension  $\alpha^{-1}(\mathcal{NG})$  is a union of two submanifolds of codimension at least  $n(n-1)$  and  $(n-1)(n-2)+1$ . Applying

Proposition 2.6.1 to each of these submanifolds in turn, we conclude that there is a residual set  $U^\ell$  of metrics  $g$  for which the map  $\mathbb{P}TM \rightarrow J^3L \oplus \mathbb{P}TM$  defined by  $V_x \mapsto (j^3g_x, V_x)$  is transverse to  $\alpha^{-1}(\mathcal{NG}^\ell)$ , and a residual set  $U^t$  of metrics such that the map is transverse to  $\alpha^{-1}(\mathcal{NG}^t)$ . The intersection of two residual sets is again residual, so  $U = U^\ell \cap U^t$  is residual. It now remains to see what it means that a metric belongs to  $U$ .

Choose some  $g \in U$ . The image of  $\rho(g) = (V_x \mapsto (j^3g_x, V_x))$  has dimension  $2n - 1$ , so in particular

$$2n - 1 < n(n - 1) \leq \text{codim}(\mathcal{NG}^t) \quad \text{when } n \geq 4.$$

Transversality then means that the image of  $\rho(g)$  is disjoint from  $\alpha^{-1}(\mathcal{NG}^t)$ . Similarly,

$$2n - 1 - \text{codim}(\mathcal{NG}^\ell) \leq 2n - 1 - (n - 1)(n - 2) - 1 \leq 0 \quad \text{if } n \geq 4,$$

so  $\text{im}(\rho(g))$  can intersect  $\mathcal{NG}^\ell$  only in submanifolds of dimension 0.

In particular, along every geodesic  $\gamma$  there is a dense set of points  $t$  where  $\dot{\gamma}(t) \notin \alpha^{-1}(\mathcal{NG})$ . By Proposition 2.6.2, this means that there is also a dense set of points where  $\dot{\gamma}$  is generic. In particular the generic condition holds.  $\square$

# A Background

## A.1 Preliminaries

### A.1.1 Manifolds and fiber bundles

**Convention A.1.1.** If nothing to the contrary is explicitly stated, all *manifolds* will be smooth, finite-dimensional, second-countable and paracompact without boundary. We will also work with manifolds with boundary, and in these instances this will be stated explicitly. We will sometimes write "manifold without boundary" for emphasis.

**Convention A.1.2.** We will use the word *submanifold* to mean the same thing as *embedded submanifold*. Similarly, "hypersurface" will mean "embedded hypersurface".

**Convention A.1.3.** The word *closed* when used about subsets, hypersurfaces or submanifolds will mean closed in the topological sense, not in the sense of compact without boundary.

**Definition A.1.4.** A subset of a smooth  $(n + 1)$ -manifold is a *topological hypersurface* if it is locally Euclidean of dimension  $n$ .

**Remark A.1.5.** Note that all subsets of a smooth manifold are Hausdorff, paracompact and second-countable, so these properties hold for all topological hypersurfaces.

**Convention A.1.6.** By *fiber bundle*, we will mean a smooth fiber bundle, i.e. a fiber bundle in which the base space, fiber and total space are smooth manifolds, and all relevant maps are smooth.

**Convention A.1.7.** When  $E_1 \xrightarrow{\pi_1} M$  and  $E_2 \xrightarrow{\pi_2} M$  are fiber bundles over a common base space we will denote the fiber bundle product of  $E_1$  and  $E_2$  by  $E_1 \oplus E_2$ . In other words,

$$E_1 \oplus E_2 = \{(e_1, e_2) \in E_1 \times E_2 \mid \pi_1(e_1) = \pi_2(e_2)\}$$

as sets.

### A.1.2 Cobordisms

**Definition A.1.8.** Let  $S$  and  $S'$  be smooth compact manifolds of dimension  $n$ . A *cobordism* between  $S$  and  $S'$  is a compact  $(n + 1)$ -dimensional manifold-with-boundary  $M$ , the boundary of which is the disjoint union  $S \sqcup S'$ . If there is a cobordism between  $S$  and  $S'$ , we say that they are *cobordant*.

The following theorem (see [24, Corollary 4.11]) characterizes the cobordism relation in terms of Stiefel-Whitney numbers.

**Theorem A.1.9.** *Two compact manifolds without boundary are cobordant if and only if their Stiefel-Whitney numbers agree.*

The cobordism relation is particularly simple in dimension 3. (See [24, p. 203].)

**Theorem A.1.10.** *Any two compact three-dimensional manifolds without boundary are cobordant.*

## A.2 Lorentzian geometry

For a more detailed discussion of the definitions in this section, see [28] and [7].

**Definition A.2.1.** A *spacetime* is a time-oriented Lorentzian manifold of dimension at least 2.

**Definition A.2.2.** A  $C^1$  submanifold  $N$  of a spacetime  $(M, g)$  is *spacelike* if the restriction of  $g$  to  $TN$  is positive definite.

**Definition A.2.3.** A  $C^1$  hypersurface  $\mathcal{H}$  in a spacetime  $(M, g)$  is a *null hypersurface* if the restriction of  $g$  to  $T_p\mathcal{H}$  is degenerate for all  $p \in \mathcal{H}$ .

**Definition A.2.4.** Let  $(M, g)$  be a spacetime. A vector  $X \in TM$  is

- *timelike* if  $g(X, X) < 0$ ,
- *lightlike* or *null* if  $g(X, X) = 0$ ,
- *causal* if  $g(X, X) \leq 0$ ,
- *spacelike* if  $g(X, X) > 0$ .

**Definition A.2.5.** A curve  $\gamma$  in a spacetime is *timelike* if for all  $t$  in the domain of  $\gamma$  the vector  $\dot{\gamma}(t)$  is timelike. A curve is *causal* if each vector  $\dot{\gamma}(t)$  is causal.

**Definition A.2.6.** Let  $M$  be a spacetime, and let  $\gamma: [a, b) \rightarrow M$  be a future-directed causal curve. A point  $p$  is a *future endpoint* of  $\gamma$  if  $p = \lim_{t \rightarrow b} \gamma(t)$ .

**Definition A.2.7.** A future-directed causal curve  $\gamma: [a, b) \rightarrow M$  is *future extendible* if there is a future-directed causal curve  $\gamma': [a, c) \rightarrow M$  with  $c > b$  and  $\gamma'|_{[a, b)} = \gamma$ .

**Definition A.2.8.** Let  $M$  be a spacetime with or without boundary. For any point  $p \in M$ , we define the *chronological future* of  $p$  to be the set

$$I^+(p) = \{q \in M : \text{there is a (non-constant) future-directed timelike curve from } p \text{ to } q\}.$$

The chronological future of a set  $A \subseteq M$  is defined to be  $I^+(A) = \bigcup_{p \in A} I^+(p)$ . Analogously, the *chronological past* of a point  $p \in M$  is the set

$$I^-(p) = \{q \in M : \text{there is a (non-constant) future-directed timelike curve from } q \text{ to } p\},$$

and the chronological past of a set  $A$  is  $I^-(A) = \bigcup_{p \in A} I^-(p)$ . We define

$$I(A) = I^+(A) \cup I^-(A).$$

We define the *causal future* and *causal past* of a point  $p$  to be

$$J^+(p) = \{q \in M \mid \text{there is a future-directed causal curve from } p \text{ to } q\}$$

and

$$J^-(p) = \{q \in M \mid \text{there is a future-directed causal curve from } q \text{ to } p\},$$

respectively.

We also define pasts and futures relative to a subset  $U \subseteq M$ . The set  $I^+(p, U)$  is the set of points which can be reached by a timelike curve which is completely contained in  $U$ . We define  $I^-(p, U)$ ,  $I^\pm(A, U)$ ,  $J^\pm(p, U)$  and  $J^\pm(A, U)$  similarly.

Note that  $I^+(A)$  and  $I^-(A)$  are open sets for each subset  $A$  of the spacetime. (See [7, Proposition 2.4.12].)

**Definition A.2.9.** Let  $M$  be a spacetime without boundary. A set  $A$  is *achronal* if no two points of  $A$  can be connected by a timelike curve. It is *acausal* if no two points can be connected by a causal curve.

**Definition A.2.10.** Let  $A$  be an achronal set. Then  $\text{edge}(A)$  denotes the set of points  $p \in \overline{A}$  such that every neighborhood  $U$  of  $p$  contains a timelike curve from  $I^-(p, U)$  to  $I^+(p, U)$  which does not intersect  $A$ .

**Definition A.2.11.** Let  $M$  be a spacetime and let  $A \subseteq M$  be an arbitrary subset. A future-directed causal curve  $\gamma: I \rightarrow M$  is *totally past imprisoned* in  $A$  if there is some  $t \in I$  such that  $\gamma(s) \in A$  for all  $s \in I$  with  $s \leq t$ .



### A.2.1 Cauchy developments and Cauchy horizons

**Definition A.2.12.** Let  $A$  be an arbitrary subset of a spacetime  $M$ . The *future Cauchy development* of  $A$  is the set

$$D^+(A) = \{p \in M \mid \text{every past inextendible causal curve through } p \text{ intersects } A\}.$$

Informally,  $D^+(A)$  consists of the points about which one has complete knowledge given initial data on  $A$ . The *past Cauchy development*  $D^-(A)$  may be defined analogously, but we will not have use for it.

Note that  $D^+(A)$  is defined in terms of causal curves. Using an analogous definition with timelike curves yields the topological closure of  $D^+(A)$ :

**Proposition A.2.13.** Let  $M$  be a spacetime without boundary, and let  $A$  be an arbitrary subset of  $M$ . Then

$$\overline{D^+(A)} = \{p \in M \mid \text{every past inextendible timelike curve through } p \text{ intersects } A\}.$$

*Proof.* See [21, Lemma 8.3.8]. □

The "future boundary" of  $D^+(A)$  will be of particular interest:

**Definition A.2.14.** Let  $A$  be an arbitrary subset of a spacetime  $M$ . The *future Cauchy horizon* of  $A$  is the set

$$H^+(A) = \overline{D^+(A)} \setminus I^-(D^+(A)).$$

**Definition A.2.15.** A topological hypersurface  $S$  in a spacetime  $M$  is a *Cauchy surface* if  $M = D^-(S) \cup D^+(S)$ .

### A.2.2 Causality conditions

**Definition A.2.16.** A spacetime is *chronological* (or "satisfies the *chronology condition*") if it contains no closed timelike curves.

**Definition A.2.17.** A spacetime is *causal* (or "satisfies the *causality condition*") if there are no closed causal curves.

**Definition A.2.18.** A spacetime is *strongly causal* (or "satisfies the *strong causality condition*") if each point has a neighborhood which intersects no timelike curve more than once.

**Definition A.2.19.** A spacetime is *stably causal* if it admits a global time function.

**Definition A.2.20.** A spacetime is *globally hyperbolic* if it contains a Cauchy surface.

**Proposition A.2.21.** *Let  $S$  be an achronal hypersurface in a spacetime  $M$ . Then the interior of  $\overline{D^+(S)} \cup \overline{D^-(S)}$  is globally hyperbolic (provided it is nonempty).*

*Proof.* See [7, Proposition 2.9.9]. □

**Proposition A.2.22.** *Let  $(M, g)$  be a spacetime. Then*

*$M$  is globally hyperbolic*

⇓

*$M$  is stably causal*

⇓

*$M$  is strongly causal*

⇓

*$M$  is causal*

⇓

*$M$  is chronological.*

### A.3 Energy conditions

Throughout this section  $(M, g)$  will denote a spacetime.  $R$  will denote its curvature tensor,  $\text{Ric}$  its Ricci curvature,  $S$  its scalar curvature, and  $T$  the *stress-energy tensor*. We define  $T$  through Einstein's equation using a cosmological constant  $\Lambda$ .

$$8\pi T = \text{Ric} + \frac{1}{2}Sg + \Lambda g.$$

We are mainly concerned with lightlike vectors, so the cosmological constant plays no role.

**Definition A.3.1.** We define the following energy conditions.

Name of energy condition	Definition
<i>Lightlike convergence condition</i>	$\text{Ric}(v, v) \geq 0$ for all lightlike vectors $v$
<i>Null energy condition</i>	$T(v, v) \geq 0$ for all lightlike vectors $v$
<i>Weak energy condition</i>	$T(v, v) \geq 0$ for all causal vectors $v$
<i>Strict lightlike convergence condition</i>	$\text{Ric}(v, v) > 0$ for all lightlike vectors $v$
<i>Strict null energy condition</i>	$T(v, v) > 0$ for all lightlike vectors $v$
<i>Strict weak energy condition</i>	$T(v, v) > 0$ for all causal vectors $v$

**Remark A.3.2.** Note that the lightlike convergence condition is equivalent to the null energy condition. Both of them are implied by the weak energy condition.

**Definition A.3.3.** Let  $M$  be a spacetime. We say that a vector  $X \in TM$  is *generic* if the tensor  $X^e X^f X_{[a} R_{b]ef[c} X_{d]}$  is not equal to zero.

**Definition A.3.4.** Let  $M$  be a spacetime with curvature tensor  $R$ . We say that  $M$  satisfies the *timelike/lightlike/spacelike generic condition* if it holds that each inextendible timelike/lightlike/spacelike geodesic  $\gamma$  has some point at which  $\dot{\gamma}(t)$  is generic.

**Definition A.3.5.** We say that a spacetime satisfies the *generic condition* if it satisfies the timelike and lightlike generic conditions.

**Remark A.3.6.** The generic condition is also known as the "genericity condition" (see [21, p. 388]) and the "generality condition" (see [23, p. 30]). The term "generic condition" is used in [19, p. 101] and [1].

## A.4 Geometric measure theory

References for this section are [13] and [25].

### A.4.1 Regularity of functions

**Definition A.4.1.** A function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is  $C^{1,1}$  if it is  $C^1$  and its differential  $df$  is Lipschitz. A submanifold of a smooth manifold is  $C^{1,1}$  if it is locally the graph of a  $C^{1,1}$  function in coordinates.

**Definition A.4.2.** A function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is *semi-convex* if it is the sum of a convex function and a  $C^2$  function. A submanifold of a smooth manifold is semi-convex if it is locally the graph of a semi-convex function in coordinates.

### A.4.2 Measure zero

Let  $\Sigma$  be some smooth manifold of dimension  $m$ , and let  $M$  be a smooth manifold of dimension at least  $m$ . Let  $\psi: \Sigma \rightarrow M$  be a topological embedding. We will consider two notions of "measure zero":

- Since  $\Sigma$  is a smooth manifold (and in particular second countable), any two Riemannian metrics on  $\Sigma$  give rise to the same family of sets having zero measure in the associated volume measure on  $\Sigma$ .

- Let  $\sigma$  be an arbitrary Riemannian metric on  $M$ . This metric induces a distance function, which in turn induces Hausdorff measures of any dimension. Let  $\mathfrak{h}^m$  denote the  $m$ -dimensional Hausdorff measure induced by  $\sigma$ . Then we say that  $A \subseteq \psi(\Sigma)$  has measure zero if  $\mathfrak{h}^m(A) = 0$ .

These two notions are connected in the following sense.

**Proposition A.4.3.** *Let  $\Sigma$  be some smooth manifold of dimension  $m$ , and let  $M$  be a smooth manifold of dimension  $n$  with  $n \geq m$ . Let  $\psi: \Sigma \rightarrow M$  be a topological embedding. Suppose that  $\psi$  is locally Lipschitz. Then  $\mathfrak{h}^m(\psi(A)) = 0$  if  $A$  has measure zero viewed as a subset of  $\Sigma$ .*

*Proof.* After representing  $\psi$  in coordinates by  $\psi: U \rightarrow V$  with open sets  $U \subseteq \Sigma$  and  $V \subseteq M$  identified with subsets of  $\mathbb{R}^m$  and  $\mathbb{R}^n$  it holds that

$$\mathfrak{h}^m(\psi(A \cap U)) \leq L\mu(A \cap U)$$

where  $L$  is the Lipschitz constant of  $\psi$  over  $U$ , and  $\mu$  denotes the  $m$ -dimensional Hausdorff measure on  $U$ . (This can be proved by bounding the volume change of images of unit balls using the Lipschitz constant, or it can be seen as a special case of the much more powerful [13, Theorem 2.10.25].) Since  $U$  is a subset of  $\mathbb{R}^m$ , this Hausdorff measure agrees up to pointwise scaling by a smooth function with the Lebesgue measure in coordinates. In particular, if  $A \cap U$  has zero measure in  $\Sigma$ , then  $\mathfrak{h}^m(\psi(A \cap U)) = 0$ . By second countability, countably many charts suffice to cover  $\psi(\Sigma)$ , and so  $\mathfrak{h}^m(\psi(A \cap U)) = 0$  if  $A$  has measure zero.  $\square$

In general, we will mostly be interested in the notions of "measure zero" and "finite measure", so it will not matter precisely which Riemannian metric is used to induce a measure.

**Proposition A.4.4.** *Let  $M$  be a smooth manifold of dimension at least  $n$  and let  $N$  be a Lipschitz submanifold of  $M$  with dimension  $n$ . Let  $K$  be a compact subset of  $N$ . Let  $\sigma$  be a Riemannian metric on  $M$ , and let  $\mathfrak{h}^n$  be the corresponding  $n$ -dimensional Hausdorff measure. Then all measurable subsets of  $K$  have finite  $\mathfrak{h}^n$ -measure.*

*Proof.* It is sufficient to show that  $K$  has finite measure, since subsets of  $K$  have smaller measure than  $K$ . Take a finite subcover of the cover  $K \subseteq \bigcup_{p \in \mathcal{H}} B_\sigma(p, 1)$ . Each  $\mathcal{H} \cap B_\sigma(p, 1)$  has finite  $\mathfrak{h}^n$ -measure since  $\mathcal{H}$  is a Lipschitz hypersurface. Hence we can conclude that  $K$  has finite measure.  $\square$

### A.4.3 Density functions

A reference for density functions is [25, Chapter 2]. We will use the same idea, but with somewhat different notation.

**Definition A.4.5.** Let  $\mathcal{L}^n$  denote Lebesgue measure on  $\mathbb{R}^n$ . For each measurable subset  $A \subseteq \mathbb{R}^n$  define the *density function of  $A$  (with respect to  $\mathcal{L}^n$ )* to be the function

$$\Theta(A, \cdot): A \rightarrow [0, 1],$$

$$\Theta(A, q) = \lim_{r \rightarrow 0} \frac{\mathcal{L}^n(A \cap B^n(q, r))}{\mathcal{L}^n(B^n(q, r))}.$$

Here  $B^n(q, r)$  denotes the ball of radius  $r$  centered at  $q$ .

**Definition A.4.6.** Let  $A$  be a measurable subset of  $\mathbb{R}^n$ . We will call the set

$$\tilde{A} = \{a \in A \mid \Theta(A, a) = 1\}$$

the *full-density subset* of  $A$ .

**Proposition A.4.7.** Let  $\tilde{A}$  be the full-density subset of some set  $A \subseteq \mathbb{R}^n$ . Then  $\tilde{A}$  has full Lebesgue measure in  $A$ .

*Proof.* By [25, Corollary 2.9] the density function  $\Theta(A, \cdot)$  is equal to the characteristic function of  $A$  almost everywhere, yielding the conclusion.  $\square$

We now generalize the notion of full-density subsets to hypersurfaces in Riemannian manifolds.

**Lemma A.4.8.** Let  $(M, \sigma)$  be a Riemannian manifold of dimension  $n + 1$  and let  $\mathcal{H}$  be a Lipschitz hypersurface. Let  $U$  be an open subset of  $\mathcal{H}$  and let  $\varphi: U \rightarrow \mathbb{R}^n$  and  $\psi: U \rightarrow \mathbb{R}^n$  be charts. Let  $A$  be a subset of  $U$  and let  $\tilde{A}_\varphi$  and  $\tilde{A}_\psi$  denote the full-density subsets of  $\varphi(A)$  and  $\psi(A)$ , respectively. Then

$$\tilde{A}_\psi = \psi(\varphi^{-1}(\tilde{A}_\varphi)).$$

*Proof.* Abbreviate  $\psi \circ \varphi^{-1}$  by  $f$ . Since  $f$  is bi-Lipschitz, it holds for any measurable subset  $X$  of  $\text{im } \varphi$  that

$$\frac{1}{L_{f^{-1}}^n} \mathcal{L}^n(X) \leq \mathcal{L}^n(f(x)) \leq L_f^n \mathcal{L}^n(X)$$

where  $L_{f^{-1}}$  and  $L_f$  denote the Lipschitz constants of  $f^{-1}$  and  $f$ . In particular,

$$\frac{\mathcal{L}^n(B^n(q, r) \setminus \varphi(A))}{\mathcal{L}^n(B^n(q, r))} \leq \frac{L_{f^{-1}}^n}{L_f^n} \frac{\mathcal{L}^n(f(B^n(q, r)) \setminus \psi(A))}{\mathcal{L}^n(f(B^n(q, r)))}.$$

By letting  $R(r)$  be a positive real number such that  $f(B^n(q, r)) \subseteq B^n(f(q), R(r))$  and  $\rho(r)$  a positive real number such that  $f(B^n(q, r)) \supseteq B^n(f(q), \rho(r))$  we see that

$$\frac{\mathcal{L}^n(B^n(q, r) \setminus \varphi(A))}{\mathcal{L}^n(B^n(q, r))} \leq \frac{L_{f^{-1}}^n}{L_f^n} \frac{\mathcal{L}^n(B^n(f(q), R(r)) \setminus \psi(A))}{\mathcal{L}^n(B^n(f(q), \rho(r)))}.$$

Since  $f$  and  $f^{-1}$  are Lipschitz, we may choose  $R$  and  $\rho$  to be bounded from above and below by linear functions of positive derivative, so there are positive constants  $D$  and  $D'$  such that

$$D\mathcal{L}^n(B^n(f(q), \rho(r))) \leq \mathcal{L}^n(B^n(f(q), R(r))) \leq D'\mathcal{L}^n(B^n(f(q), \rho(r))).$$

This together with the previous inequality means that there is a positive real number  $C$  independent of  $r$  such that

$$\frac{\mathcal{L}^n(B^n(q, r) \setminus \varphi(A))}{\mathcal{L}^n(B^n(q, r))} \leq C \frac{\mathcal{L}^n(B^n(f(q), R(r)) \setminus \psi(A))}{\mathcal{L}^n(B^n(f(q), R(r)))}.$$

In particular, if  $\Theta(\psi(A), f(q)) = 1$  so that

$$\lim_{r \rightarrow 0} \frac{\mathcal{L}^n(B^n(f(q), R(r)) \setminus \psi(A))}{\mathcal{L}^n(B^n(f(q), R(r)))} = 0$$

then

$$\lim_{r \rightarrow 0} \frac{\mathcal{L}^n(B^n(q, r) \setminus \varphi(A))}{\mathcal{L}^n(B^n(q, r))} = 0$$

so that  $\Theta(\phi(A), q) = 1$ . Hence

$$\Theta(\psi(A), f(q)) = 1 \implies \Theta(\phi(A), q) = 1.$$

Repeating this argument for the inverse of  $f$  we see that

$$\Theta(\phi(A), q) = 1 \implies \Theta(\psi(A), f(q)) = 1.$$

This proves that  $\tilde{A}_\psi = f(\tilde{A}_\phi)$ , completing the proof.  $\square$

**Definition A.4.9.** Consider a Riemannian manifold  $(M, \sigma)$  of dimension  $n + 1$  and let  $\mathcal{H}$  be a Lipschitz hypersurface. In light of the previous proposition, we may define the *full-density subset* of a set  $A \subseteq \mathcal{H}$  to be the set  $\tilde{A}$  such that if  $\varphi: U \rightarrow \mathbb{R}^n$  is a chart on  $\mathcal{H}$  then  $\varphi(\tilde{A} \cap U)$  is the full-density subset of  $\varphi(A \cap U)$ .

**Definition A.4.10.** If  $q$  belongs to the full-density subset of a set  $A$ , we say that  $q$  is a *full-density point* of  $A$ .

**Proposition A.4.11.** Let  $(M, \sigma)$  be a Riemannian manifold of dimension  $n + 1$  and let  $\mathcal{H}$  be a Lipschitz hypersurface. Let  $\mathfrak{h}^n$  be the  $n$ -dimensional Hausdorff measure constructed from  $\sigma$ . Let  $A \subseteq \mathcal{H}$  be any subset and let  $\tilde{A}$  be its full-density subset. Then  $\mathfrak{h}^n(A \setminus \tilde{A}) = 0$ .

*Proof.* Since  $\mathcal{H}$  is second-countable, it is sufficient to prove this locally. This can be done by the use of charts and Proposition A.4.7.  $\square$

**Proposition A.4.12.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  and let  $f, g: \Omega \rightarrow \mathbb{R}$  be Lipschitz functions. Let  $A \subset \Omega$  be a measurable subset of  $\Omega$  and suppose that  $f$  and  $g$  agree on  $A$ . Let  $q$  be a full-density point of  $A$  and suppose that  $f$  and  $g$  are both differentiable at  $q$ . Then  $df(q) = dg(q)$ .*

*If, moreover,  $q$  is a point where  $f$  and  $g$  have second order expansions of the form*

$$f(q + \xi) = f(q) + df(q)(\xi) + \frac{1}{2}D^2f(q)(\xi, \xi),$$

$$g(q + \xi) = g(q) + dg(q)(\xi) + \frac{1}{2}D^2g(q)(\xi, \xi),$$

*then  $D^2f(q) = D^2g(q)$ .*

*Proof.* Let  $h = f - g$  and note that  $h$  is differentiable at  $q$  and zero on  $A$ . Suppose that  $dh(q)(V) \neq 0$  for some vector  $V$  at  $q$ . By continuity  $dh(q)(W) \neq 0$  for all  $W$  in some open neighborhood  $U$  of  $V$  in  $T_q\mathbb{R}^n$ . Then for all sufficiently small  $\epsilon > 0$  and all  $W \in U$  it holds that  $h(q + \epsilon W) \neq 0$ . This means that  $h$  is nonzero on some small open cone in the direction of  $V$ , which in turn means that  $\Theta(A, q)$  cannot be equal to 1, contradicting the fact that  $q$  is a full-density point of  $A$ . This shows that  $dh(q) = 0$ , proving the first part of the proposition.

Suppose now that  $q$  is a point where  $f$  and  $g$  have second order expansions in the sense that

$$f(q + \xi) = f(q) + df(q)(\xi) + \frac{1}{2}D^2f(q)(\xi, \xi),$$

$$g(q + \xi) = g(q) + dg(q)(\xi) + \frac{1}{2}D^2g(q)(\xi, \xi).$$

Then, since  $f(q) = g(q)$  and  $df(q) = dg(q)$ ,

$$f(q + \xi) - g(q + \xi) = \frac{1}{2}(D^2f(q) - D^2g(q))(\xi, \xi).$$

If  $(D^2f(q) - D^2g(q))(\xi, \xi) = 0$  for every vector  $\xi$  then  $D^2f(q) - D^2g(q) = 0$  and we are done (since  $D^2f(q) - D^2g(q)$  is symmetric), so suppose that there is some  $\xi$  with  $(D^2f(q) - D^2g(q))(\xi, \xi) \neq 0$ . By continuity, this then holds for all  $v$  in a neighborhood of  $\xi$ , and this means that  $f - g$  is nonzero on a small open cone from  $q$ . This means that  $\Theta(A, q)$  cannot be equal to 1 contradicting the fact that  $q$  is a full-density point of  $A$ , showing that indeed  $D^2f(q) = D^2g(q)$ .  $\square$

## A.5 Jet bundles

In Chapter 2 we need the concept of "jets". In this section, we give the necessary definitions. For the most part, we use the notation of [31]. For jets of arbitrary maps rather than jets of sections of a smooth fiber bundle, we follow [17, §2 Chapter II]. The idea of jets is to capture the concept of an "abstract Taylor polynomial" of a function between manifolds, by considering equivalence classes of functions which have the same Taylor polynomials in coordinates.

### A.5.1 Jets of maps between manifolds

**Definition A.5.1.** Let  $X$  and  $Y$  be smooth manifolds, and choose a nonnegative integer  $k$ . Let  $P \subseteq X \times \mathcal{P}(X)$  denote the set of those pairs  $(x, U)$  such that  $x \in U$ . Let  $J^k(X, Y)$  be the set of equivalence classes on the set  $P \times C^\infty(X, Y)$  under the relation defined by  $(x_1, U_1, f_1) \sim (x_2, U_2, f_2)$  if

- $x_1 = x_2$ , and
- $f_1|_{U_1} = f_2|_{U_2}$ , and
- after choosing coordinates around  $x_1 = x_2$  and  $f_1(x_1) = f_2(x_2)$ , the partial derivatives of  $f_1$  and  $f_2$  at  $x_1 = x_2$  agree up to order  $k$ .

This is independent of the coordinates chosen. The set  $J^k(X, Y)$  is called the set of  $k$ -jets of maps from  $X$  to  $Y$ .

Locally,  $J^k(X, Y)$  looks like a product  $U \times V \times \mathbb{R}^m$  (where  $U \subseteq X$ ,  $V \subseteq Y$ ), and this local structure can be patched together to make  $J^k(X, Y)$  into a smooth fiber bundle over  $X \times Y$  (see [17, Theorem 2.7, Chapter II]).

One can define several useful maps on jet bundles.

**Definition A.5.2.** Let  $X$  and  $Y$  be smooth manifolds and let  $k$  be a nonnegative integer. Define the *source map*  $\pi^s: J^k(X, Y) \rightarrow X$  by

$$\pi^s([x, f]) = x$$

and the *target map*  $\pi^t: J^k(X, Y) \rightarrow Y$  by

$$\pi^t([x, f]) = f(x).$$

Let  $0 \leq l \leq k$  and define the  $l$ -jet projection  $\pi_l: J^k(X, Y) \rightarrow J^l(X, Y)$  by

$$\pi_l([x, f]) = [x, f].$$



Note that two different equivalence relations are used in the preceding equation, and that  $\pi_l$  is well-defined since the equivalence relation defining  $J^l(X, Y)$  is coarser than the one defining  $J^k(X, Y)$ . For each point  $x \in X$ , there is a canonical map  $j_x^k: C^\infty(X, Y) \rightarrow J^k(X, Y)$  defined by  $f \mapsto [x, f]$ . We will call  $j_x^k f$  the *1-jet of  $f$  at  $x$* . We will sometimes denote this by  $j^k f(x) := j_x^k f$ , and call  $j^k f$  the *1-jet of  $f$* .

### A.5.2 Jets of sections of a smooth fiber bundle

To define jets of sections of a smooth fiber bundle one proceeds similarly.

**Definition A.5.3.** Let  $E \rightarrow B$  be a smooth fiber bundle, and let  $k$  be a nonnegative integer. Let  $J^k E$  be the set of equivalence classes on the set  $X \times \Gamma^\infty(E)$  of sections, under the relation defined by  $(x_1, \sigma_1) \sim (x_2, \sigma_2)$  if

- $x_1 = x_2$ , and
- $\sigma_1(x_1) = \sigma_2(x_2)$ , and
- after choosing coordinates for a local trivialization of  $E$  around  $x_1 = x_2$  and  $\sigma_1(x_1) = \sigma_2(x_2)$ , and locally viewing the sections  $\sigma_i$  as functions  $f_i$  from the trivializing neighborhoods to the fiber, the partial derivatives of  $f_1$  and  $f_2$  at  $x_1 = x_2$  agree up to order  $k$ .

We may define the source, target and  $l$ -jet projection maps as in the previous section. Note that for sections of a bundle, there is an identification of  $J^0 E$  with  $E$  itself, and under this identification the 0-jet projection  $\pi_0: J^k E \rightarrow J^0 E$  agrees with the target map  $\pi^t: J^k E \rightarrow E$ . As was the case for the space of jets of functions, the space  $J^k E$  has a natural structure as a fiber bundle over  $B$ .

## A.6 Whitney topologies

**Definition A.6.1.** A subset of a topological space is *residual* if it is a countable intersection of open dense sets. A topological space is a *Baire space* if every residual set is dense.

Let  $X$  and  $Y$  be smooth manifolds. We will define a family of topologies on  $C^\infty(X, Y)$  called the *Whitney  $C^k$  topologies*. We will follow [17, §3 Chapter II].

**Definition A.6.2.** Let  $X$  and  $Y$  be smooth manifolds. Fix some nonnegative integer  $k$ . For each open subset  $U$  of  $J^k(X, Y)$ , denote

$$M^k(U) = \left\{ g \in C^\infty(X, Y) \mid j^k g(X) \subseteq U \right\}.$$

The collection

$$\{M^k(U) \mid U \subseteq J^k(X, Y) \text{ is open}\}$$

now forms a basis for a topology on  $C^\infty(X, Y)$ . We call this topology the *Whitney  $C^k$  topology*.

**Remark A.6.3.** This topology is also known as the *strong* or *fine  $C^k$  topology*.

**Definition A.6.4.** Let  $X$  and  $Y$  be smooth manifolds. The collection

$$\{M^k(U) \mid k \geq 0 \text{ and } U \subseteq J^k(X, Y) \text{ is open}\}$$

forms a basis for a topology on  $C^\infty(X, Y)$ . We call this topology the *Whitney  $C^\infty$  topology*.

We need the fact that the Whitney  $C^\infty$  makes  $C^\infty(X, Y)$  a Baire space. A proof of this fact can be found in [17, Proposition 3.3 in Chapter II].

**Proposition A.6.5.** *Let  $X$  and  $Y$  be smooth manifolds. The space  $C^\infty(X, Y)$  endowed with the Whitney  $C^\infty$  topology is a Baire space.*

**Corollary A.6.6.** *Let  $X$  and  $Y$  be smooth manifolds. The space  $C^\infty(X, Y)$  endowed with the Whitney  $C^r$  topology for any  $r \geq 1$  is a Baire space.*

# Assorted proofs

## B.1 Complete generators

The following is a straightforward generalization of Lemma 8.5.5 in [19], and the proof follows that of [19] but contains significantly more details.

**Lemma B.1.1.** *Let  $S$  be an achronal hypersurface in a spacetime  $(M, g)$ . Let  $\gamma$  be a null geodesic segment contained  $H^+(S)$ . Suppose that  $\gamma$  has no past endpoint and is totally past imprisoned in some compact set  $K$ . Then  $\gamma$  is complete in the past direction.*

*Proof.* Let  $\gamma$  have an affine parametrization. Suppose to get a contradiction that  $\gamma$  is incomplete to the past, i.e. that the domain of  $\gamma$  has some infimum  $\nu_0$ . We may without loss of generality (by translation of the parameter of  $\gamma$  and restriction of  $\gamma$  to a smaller domain) assume that  $\gamma$  has domain  $(\nu_0, 0]$  and that  $\gamma(t) \in K$  for all  $t \in (\nu_0, 0]$ .

The idea is now to show that if  $\gamma$  is past incomplete, then a small perturbation of it yields a past inextendible timelike curve with contradictory properties. To help with this, we introduce an auxiliary Riemannian metric: Let  $\mathcal{W}$  be an open neighborhood of  $H^+(S) \cap K$  with compact closure. Let  $V$  be a future-directed unit timelike vector field on  $M$  such that  $\nabla_V V = 0$  everywhere on  $\mathcal{W}$ . Define a metric  $g'$  by

$$g'(X, Y) = g(X, Y) + 2g(X, V)g(Y, V).$$

This metric is positive definite. To see this, let  $X$  be nonzero and compute  $g'(X, X)$  in a basis  $(V, e_1, e_2, e_3)$  orthonormal for  $g$ :

$$g'(X, X) = g(X, X) + 2(g(X, V))^2 = -(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + 2(X^0)^2 > 0.$$

Let  $\alpha_0(t) = \gamma(\nu(t))$  be a reparametrization of  $\gamma$  such that  $g(\dot{\alpha}_0, V) = -1/\sqrt{2}$ . This implies that  $\nu$  is strictly increasing. For convenience, suppose also that  $\nu(0) =$

0. With this choice, the curve length in  $g'$  of  $\alpha_0$  on the parameter interval  $[a, b]$  is

$$\begin{aligned} \int_a^b \sqrt{g'(\dot{\alpha}_0(t), \dot{\alpha}_0(t))} dt &= \int_a^b \sqrt{g(\dot{\alpha}_0(t), \dot{\alpha}_0(t)) + 2(g(\alpha_0(t), V))^2} dt \\ &= \int_a^b \sqrt{0 + 2 \frac{1}{2}} dt = b - a \end{aligned}$$

so  $\alpha_0$  is parameterized by arc length in the Riemannian metric  $g'$ . Since  $\gamma$  has no past endpoint,  $\alpha_0$  does not have one either.

We will later construct a variation  $\alpha$  of  $\alpha_0$ , and the computations will be done along the two-parameter map  $\alpha$ .

**Claim I: The domain of  $\alpha_0$  is not bounded from below**

Suppose for contradiction that the domain of  $\alpha_0$  were bounded below. Let  $a > -\infty$  be the infimum of the domain of  $\alpha_0$ . Recall that a Riemannian metric induces a distance function defined as the infimum of the lengths of curves from one point to the other. Then for any sequence  $a_n \rightarrow a$  it holds that  $\alpha_0(a_n)$  is a Cauchy sequence with respect to the distance function induced by  $g'$  (for  $\alpha_0$  is a curve of length  $|a_n - a_m| < |a_{\min(m,n)} - a| \rightarrow 0$  connecting  $\alpha_0(a_n)$  to  $\alpha_0(a_m)$ ). The sequence  $\alpha_0(a_n)$  is also contained in the compact set  $K$ , and so has a convergent subsequence. These two statements together imply that  $\alpha_0(a_n)$  is convergent for any sequence  $a_n \rightarrow a$  so the limit  $\lim_{t \rightarrow a^+} \alpha_0(t)$  exists contradicting the fact that  $\alpha_0$  has no past endpoint. Hence the domain of  $\alpha_0$  is not bounded from below.

**Proof part II (Relations between  $\alpha_0$  and  $\gamma$ ):** Since  $\alpha_0$  is a reparametrization of a geodesic,  $\nabla_{\dot{\alpha}_0} \dot{\alpha}_0$  is parallel to  $\dot{\alpha}_0$ . In other words, there is a function  $f: (-\infty, 0) \rightarrow \mathbb{R}$  such that

$$\nabla_{\dot{\alpha}_0(t)} \dot{\alpha}_0(t) = f(t) \dot{\alpha}_0(t), \quad \forall t \in (-\infty, 0).$$

It also holds that

$$v'(t) \dot{\gamma}(v(t)) = \dot{\alpha}_0(t), \quad \forall t \in (-\infty, 0).$$

Now

$$f(t) \dot{\alpha}_0(t) = \nabla_{\dot{\alpha}_0} \dot{\alpha}_0 = \nabla_{\dot{\alpha}_0} (v' \dot{\gamma}) = \alpha_0(v') \dot{\gamma} + v' \nabla_{\dot{\gamma}} \dot{\gamma} = \frac{v''(t)}{v'(t)} \dot{\alpha}_0(t)$$

so

$$f = \frac{v''}{v'}.$$

Note also that  $f$  is bounded. This can be seen by computing the norm of  $f \dot{\alpha}_0$  in the metric  $g'$ .

$$\begin{aligned} f &= -\sqrt{2} g(f \dot{\alpha}_0, V) = -\sqrt{2} g(\nabla_{\dot{\alpha}_0} \dot{\alpha}_0, V) = -\sqrt{2} (\nabla_{\dot{\alpha}_0} g(\dot{\alpha}_0, V) - g(\dot{\alpha}_0, \nabla_{\dot{\alpha}_0} V)) \\ &= -\sqrt{2} (\dot{\alpha}_0(-1/\sqrt{2}) - g(\dot{\alpha}_0, \nabla_{\dot{\alpha}_0} V)) = \sqrt{2} g(\dot{\alpha}_0, \nabla_{\dot{\alpha}_0} V). \end{aligned}$$

This shows that  $f$  can be defined in terms of  $g$ ,  $\dot{\alpha}$  and  $V$ . The coefficients of these in coordinate patches are all bounded since  $\dot{\alpha}$  is a unit vector field in  $g'$ . Since  $H^+(S) \cap K$  is compact it can be covered by finitely many coordinate patches, and hence  $f$  is bounded.

**Claim III:  $v'$  is bounded**

Since  $\gamma$  is incomplete to the past,  $v$  is bounded below. In other words, the integral

$$v(t) = \int_0^t v'(\tau) d\tau$$

is bounded. This of course implies that  $\liminf_{t \rightarrow -\infty} v'(t) = 0$ , since  $v$  is strictly increasing. We will now show that boundedness of  $v$  on  $(-\infty, 0]$  together with boundedness of  $f = \frac{v''}{v'}$  implies that  $v'$  is bounded. Suppose not. Since  $v'$  is continuous, it can only be unbounded on  $(-\infty, 0]$  if  $\limsup_{t \rightarrow -\infty} v'(t) = \infty$ . Since we also know that  $\liminf_{t \rightarrow -\infty} v'(t) = 0$  and that  $v'$  is continuous there are, for arbitrarily large  $C > 0$ , sequences  $t_n, s_n \rightarrow -\infty$  such that

$$t_{n+1} < s_n < t_n \text{ for all } n,$$

$$v'(t_n) = 2C,$$

$$v'(s_n) = C$$

and

$$C \leq v'(t) \leq 2C \text{ if } t \in (s_n, t_n).$$

By the mean value theorem of calculus, there is for each  $n$  some  $\tau_n \in [s_n, t_n]$  such that

$$v''(\tau_n) = \frac{v'(t_n) - v'(s_n)}{t_n - s_n} = \frac{C}{t_n - s_n}.$$

However

$$\sum_{n=0}^{\infty} C(t_n - s_n) \leq \left| \int_0^{-\infty} v'(\tau) d\tau \right| < \infty$$

so  $(t_n - s_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Hence

$$\lim_{n \rightarrow \infty} f(\tau_n) v'(\tau_n) = \lim_{n \rightarrow \infty} v''(\tau_n) = \infty.$$

Since  $v'(\tau_n) \in [C, 2C]$  for all  $n$ , this implies that  $f(\tau_n) \rightarrow \infty$ , contradicting the fact that  $f$  is bounded. Hence  $v'$  must be bounded.

**Proof part IV(Construction of a variation  $\alpha$  of  $\alpha_0$ ):** We will now construct a variation  $\alpha$  of  $\alpha_0$ . The idea is to push  $\alpha_0$  to the past and make it timelike, and then derive a contradiction from the resulting curve. Let  $x: (-\infty, 0) \rightarrow \mathbb{R}$  denote a smooth positive function which will be fixed later. Let

$$\alpha: (-\delta, \delta) \times (-\infty, 0) \rightarrow H^+(S)$$

$$(u, t) \mapsto \alpha(u, t)$$

be a smooth map such that

$$\alpha(0, \cdot) = \alpha_0 \text{ and } \frac{\partial \alpha}{\partial u}(u, t) = -x(t) V_{\alpha(u, t)}. \quad (\text{B.1})$$

To see that such a variation exists, note that the conditions can be viewed as a family of ordinary differential equations in  $u$ , parameterized by  $t$ . This means that a smooth solution exists for each  $t$  for some small  $\delta_t > 0$  (depending on  $t$ ) by the existence theorem and theorem about smooth dependence on initial values for ordinary differential equations. To claim that the necessary variation exists, we need to show that the existence times  $\delta_t$  can be uniformly bounded from below by some  $\delta > 0$  independent of  $t$ . However, we know that a solution to the differential equation exists as long as it stays in the compact set  $\overline{\mathcal{W}}$ . Since  $H^+(S) \cap K$  is compact and  $\mathcal{W}$  open, the  $g'$  distance between  $H^+(S) \cap K$  and  $M \setminus \mathcal{W}$  is positive. Since  $V$  is bounded, and  $x$  will be bounded when we choose it, there is a positive uniform lower bound for the time after which a solution may leave  $\mathcal{W}$ . This means that there is a uniform lower bound for the existence times of the solutions of the family of ordinary differential equations defining the variation. Hence a variation with the desired properties exists. When we later fix the function  $x$ , it will be bounded. Since  $H^+(S) \cap K$  is compact and contained in the open set  $\mathcal{W}$ , the distance between  $H^+(S) \cap K$  and  $M \setminus \mathcal{W}$  in the metric  $g'$  is positive. This means (since  $g'(V, V)$  is bounded) that there is some  $\delta > 0$  such that the image of  $\alpha$  is contained in  $\mathcal{W}$ . We will choose such a value for  $\delta$ .

For ease of notation, let  $\alpha_u$  denote the curve  $\alpha(u, \cdot)$ . We now wish to choose the (positive) function  $x$  in such a way that some curve  $\alpha_\epsilon$  is timelike. In other words, we want there to be some  $\epsilon > 0$  such that the function

$$y(u, t) = g(\dot{\alpha}_u(t), \dot{\alpha}_u(t))$$

is negative for  $u = \epsilon$  and all  $t \in (-\infty, 0)$ . To show that this is the case, we will compute  $\frac{\partial y}{\partial u} \Big|_{u=0}$  and a bound for  $\frac{\partial^2 y}{\partial u^2}$ , and from this obtain an upper bound for  $y$ . Choosing a suitable function  $x$  will make this upper bound negative for small values of  $u$ .

**Proof part V(Computation of  $\frac{\partial y}{\partial u}$ ):** Let  $U$  denote the pushforward through  $\alpha$  of the coordinate vector field  $\frac{\partial}{\partial u}$  on  $(-\delta, \delta) \times (-\infty, 0)$ . We will not always write out the dependence on  $t$  and  $u$ . The first partial derivative of  $y$  can be computed as

$$\begin{aligned} \frac{\partial y}{\partial u}(u, t) &= \frac{\partial}{\partial u} g(\dot{\alpha}_u(t), \dot{\alpha}_u(t)) = U g(\dot{\alpha}_u, \dot{\alpha}_u) = 2g(\nabla_U \dot{\alpha}_u, \dot{\alpha}_u) \\ &= 2g(\nabla_{\dot{\alpha}_u} U, \dot{\alpha}_u) = 2(\nabla_{\dot{\alpha}_u} g(U, \dot{\alpha}_u) - g(U, \nabla_{\dot{\alpha}_u} \dot{\alpha}_u)) \end{aligned}$$

where  $\nabla_U \dot{\alpha}_u = \nabla_{\dot{\alpha}_u} U$  since  $U$  and  $\dot{\alpha}_u$  are pushforwards of coordinate vector fields. Evaluating this at  $u = 0$  we see that

$$\begin{aligned}
\frac{\partial y}{\partial u}(0, t) &= 2(\nabla_{\dot{\alpha}_0} g(-xV, \dot{\alpha}_0) - g(-xV, \nabla_{\dot{\alpha}_0} \dot{\alpha}_0)) \\
&= 2(-\nabla_{\dot{\alpha}_0}(xg(V, \dot{\alpha}_0)) + xg(V, \nabla_{v'\dot{\gamma}}(v'\dot{\gamma}))) \\
&= 2\left(\frac{1}{\sqrt{2}}\dot{\alpha}_0(x) + xv'g(V, \nabla_{\dot{\gamma}}(v'\dot{\gamma}))\right) \\
&= 2\left(\frac{1}{\sqrt{2}}\dot{\alpha}_0(x) + x(v')^2g(V, \nabla_{\dot{\gamma}}(\dot{\gamma})) + \frac{x}{v'}\dot{\alpha}_0(v')g(V, \dot{\alpha}_0)\right) \\
&= \sqrt{2}x'(t) - \sqrt{2}\frac{x(t)v''(t)}{v'(t)} \\
&= \sqrt{2}v'(t)\frac{d}{dt}\left(\frac{x(t)}{v'(t)}\right),
\end{aligned}$$

where we have used that  $\dot{\alpha}_0 = v'\dot{\gamma}$ ,  $g(V, \dot{\alpha}_0) = -1/\sqrt{2}$  and  $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ .

**Proof part VI(An upper bound for  $\frac{\partial^2 y}{\partial u^2}$ ):** We now compute an upper bound for the second partial derivative of  $y$  with respect to  $u$ . For convenient notation, we use the vector fields

$$\begin{aligned}
T &= \alpha^*\left(\frac{\partial}{\partial t}\right), \\
U &= \alpha^*\left(\frac{\partial}{\partial u}\right).
\end{aligned}$$

Note that

$$T_{\alpha(u,t)} = \dot{\alpha}_u(t)$$

and

$$U_{\alpha(u,t)} = -x(t)V_{\alpha(u,t)}.$$

Now

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2}{\partial u^2}y(u, t) &= \frac{1}{2}\frac{\partial^2}{\partial u^2}g(\dot{\alpha}_u(t), \dot{\alpha}_u(t)) = \frac{1}{2}\frac{\partial^2}{\partial u^2}g(T, T) = \frac{\partial}{\partial u}g(\nabla_U T, T) \\
&= g(\nabla_U T, \nabla_U T) + g(\nabla_U \nabla_U T, T) = g(\nabla_T U, \nabla_T U) + g(\nabla_U \nabla_T U, T) \\
&= g(\nabla_T U, \nabla_T U) + g(\nabla_T \nabla_U U, T) + g(R(U, T)U, T)
\end{aligned}$$

where we have used that  $\nabla_U T = \nabla_T U$  since  $U$  and  $T$  are coordinate vector fields and

$$\nabla_U \nabla_K = \nabla_K \nabla_U + R(U, K).$$

We now compute each term separately.

Evaluating the first term at  $\alpha(0, t)$  and using that  $T(x) = \alpha_0(x) = x'$  we get

$$g(\nabla_T U, \nabla_T U) = g(\nabla_T(xV), \nabla_T(xV)) = g(T(x)V + x\nabla_T V, T(x)V + x\nabla_T V)$$

$$\begin{aligned}
&= (x'(t))^2 g(V, V) + 2x(t)x'(t)g(V, \nabla_T V) + x^2(t)g(\nabla_T V, \nabla_T V) \\
&= -(x'(t))^2 + (x(t))^2 g(\nabla_T V, \nabla_T V).
\end{aligned}$$

We have used that  $g(V, \nabla_T V) = 0$ . That this is true is seen by noting that

$$g(V, \nabla_T V) = Tg(V, V) - g(\nabla_T V, V) = T(-2^{-1/2}) - g(V, \nabla_T V) = -g(V, \nabla_T V)$$

so that  $g(V, \nabla_T V) = -g(V, \nabla_T V)$ . For the second term, note that

$$\nabla_U U = \nabla_U(-xV) = xU(x)V + x^2\nabla_V V = x(t)\frac{\partial x}{\partial u}V + 0 = 0$$

(since  $\nabla_V V = 0$  in the neighborhood  $\mathcal{W}$  by choice of  $V$ , and  $x$  is independent of  $u$ ) so that

$$g(\nabla_T \nabla_U U, T) = g(\nabla_T 0, T) = 0.$$

The third term is simply

$$g(R(U, T)U, T) = g(R(-xV, T)(-xV), T) = x^2(t)g(R(V, T)V, T).$$

Hence

$$\begin{aligned}
\frac{1}{2} \frac{\partial^2}{\partial u^2} g(\dot{\alpha}_u(t), \dot{\alpha}_u(t)) &= -(x'(t))^2 + (x(t))^2 (g(\nabla_T V, \nabla_T V) + g(R(V, T)V, T)) \\
&\leq x^2 (g(\nabla_T V, \nabla_T V) + g(R(V, T)V, T)).
\end{aligned}$$

We wish to bound this by  $C^2 x^2 g'(T, T)$  for some constant  $C$  on the neighborhood  $\mathcal{W}$  of  $H^+(S)$ , which we chose to have compact closure. To see that this is possible, view  $g(\nabla_T V, \nabla_T V) + g(R(V, T)V, T)$  as a quadratic form in  $T$ . Its components in coordinates depend on  $g$ ,  $V$  and  $R$ , all of which are bounded in coordinate neighborhoods, and  $H^+(S) \cap K$  can be covered by finitely many such neighborhoods. Since the quadratic form  $g'$  is positive definite, there is some  $C$  such that  $g(\nabla_T V, \nabla_T V) + g(R(V, T)V, T) \leq g'(T, T)$ . Hence

$$\frac{\partial^2}{\partial u^2} g(\dot{\alpha}_u(t), \dot{\alpha}_u(t)) \leq C^2 x^2 g'(T, T)$$

for some constant  $C$ . We want a bound in terms of  $g(\dot{\alpha}_u(t), \dot{\alpha}_u(t))$  instead, so we compute

$$g'(T, T) = g(T, T) + 2(g(V, T))^2.$$

Since

$$\begin{aligned}
\frac{\partial}{\partial u} g(V, T) &= Ug(V, T) = g(-x\nabla_V V, T) + g(V, \nabla_U T) = 0 + g(V, \nabla_T U) \\
&= -g(V, T(x)V - x\nabla_T V) = T(x) + xg(V, \nabla_T V) = x'(t)
\end{aligned}$$



(where as earlier  $g(V, \nabla_T V) = 0$ ) we know that

$$g(V, T) = ux'(t) + g(V, T)|_{u=0} = ux'(t) - \frac{1}{\sqrt{2}}.$$

When we choose  $x$ , we will make sure that  $\frac{dx}{dt}$  is bounded, and then  $2(g(V, T))^2$  is bounded by some constant  $d$  for all small  $u$ . Hence we can convert our bound in terms of  $g'(T, T)$  to a bound in terms of  $g(T, T)$ :

$$\frac{\partial^2}{\partial u^2} g(\dot{\alpha}_u(t), \dot{\alpha}_u(t)) \leq C^2 x^2 g'(T, T) \leq C^2 x^2 (g(T, T) + d).$$

In the notation of the function  $y$ , we now know that

$$\frac{\partial^2 y}{\partial u^2}(u, t) \leq (y(u, t) + d)C^2(x(t))^2.$$

**Claim VII: For all sufficiently small  $\epsilon > 0$ , the curve  $\alpha_\epsilon$  is timelike**

From our previous calculation we know that

$$\frac{\partial y}{\partial u}(0, t) = \frac{v'(t)}{\sqrt{2}} \frac{d}{dt} \left( \frac{x(t)}{v'(t)} \right).$$

Moreover,  $y(0, t) = 0$  since  $\alpha_0$  is a lightlike curve. For each fixed  $t$ , this is a differential inequality in the variable  $u$ . Let  $z$  be the solution of the differential equation resulting from replacing the inequality with equality:

$$\frac{\partial^2 z}{\partial u^2}(u, t) = C^2 x^2(0, t)(z(u, t) + d),$$

$$\frac{\partial z}{\partial u}(0, t) = \frac{\partial y}{\partial u}(0, t),$$

$$z(0, t) = y(0, t) = 0.$$

Integrating the inequality  $\frac{\partial^2 y}{\partial u^2}(u, t) \leq \frac{\partial^2 z}{\partial u^2}(u, t)$  we see that

$$\frac{\partial y}{\partial u}(u, t) - \frac{\partial y}{\partial u}(0, t) \leq \frac{\partial z}{\partial u}(u, t) - \frac{\partial z}{\partial u}(0, t)$$

so that

$$\frac{\partial y}{\partial u}(u, t) \leq \frac{\partial z}{\partial u}(u, t).$$

Integrating once again and using the fact that  $z(0, t) = y(0, t) = 0$  we have

$$y(u, t) \leq z(u, t).$$

Solving the differential equation for  $z$  we see that

$$z(u, t) = d \cosh(Cx(t)u) + a(t) \sinh(Cx(t)u) - d$$

where

$$a(t) = \frac{v'(t)}{\sqrt{2}Cx(t)} \frac{d}{dt} \left( \frac{x(t)}{v'(t)} \right).$$

Since  $d$  is nonnegative, an upper bound for  $z$  is

$$z(u, t) = d \cosh(Cx(t)u) + a(t) \sinh(Cx(t)u) - d$$

$$= (d \tanh(Cx(t)u) + a(t)) \sinh(Cx(t)u) - d \leq (d \tanh(Cx(t)u) + a(t)) \sinh(Cx(t)u).$$

Hence

$$y(u, t) \leq (d \tanh(Cx(t)u) + a(t)) \sinh(Cx(t)u).$$

Recall that the idea was to choose the function  $x$  in such a way that there exists some  $\epsilon > 0$  such that  $y(\epsilon, t) < 0$  for all  $t$ . We claim that a possible choice of  $x$  is

$$x(t) = \frac{v'(t)}{v(t) - 2v_0}.$$

Recall that

$$v_0 = \lim_{t \rightarrow -\infty} v(t)$$

and that  $v$  is increasing so that

$$v_0 \leq v(t) \leq 0 \quad \forall t \in (-\infty, 0].$$

We begin by making good on the promises we made about the function  $x$ : It should be positive, bounded, and have bounded derivative. The denominator in the definition of  $x$  is bounded from below by  $-\nu_0$  and from above by  $-2\nu_0$ , and  $-\nu_0$  is positive, so boundedness and positivity of  $x$  follow from boundedness and positivity of  $v'$ . Computing the derivative of  $x$  we see that

$$x'(t) = \frac{v''(t)}{v(t) - 2v_0} - \frac{(v'(t))^2}{(v(t) - 2v_0)^2} = \frac{v'(t)f(t)}{v(t) - 2v_0} - x^2(t)$$

Since  $x$ ,  $v'$  and  $f = v''/v'$  are bounded, so is  $x'$ . Having chosen  $x$ , we can now fix the number  $\delta > 0$  defining the domain of  $\alpha$  such that the image of  $\alpha$  is contained in  $\mathcal{W}$ .

Recall that

$$y(u, t) \leq (d \tanh(Cx(t)u) + a(t)) \sinh(Cx(t)u)$$

where

$$a(t) = \frac{v'(t)}{\sqrt{2}Cx(t)} \frac{d}{dt} \left( \frac{x(t)}{v'(t)} \right).$$

With our present choice of  $x$ ,

$$a(t) = -\frac{x(t)}{\sqrt{2}Cv'(t)}.$$

The objective is to ensure that  $y(u, t) < 0$  for some positive  $u$  and for all  $t$ . Since  $\sinh(Cxu) \geq 0$  for positive  $u$ , a sufficient condition is that

$$d \tanh(Cx(t)u) - \frac{x(t)}{\sqrt{2}Cv'(t)} < 0$$

for some  $u > 0$  and all  $t$ . A series expansion tells us that

$$\tanh(Cx(t)u) = Cx(t)u + \mathcal{O}((ux(t))^3)$$

so that

$$d \tanh(Cx(t)u) + a(t) = \left( dCu - \frac{1}{\sqrt{2}Cv'(t)} \right) x(t) + \mathcal{O}(u^3 x^3(t)).$$

Since  $v'$  is bounded, there is some positive lower bound for  $1/v'$ . Hence it holds for all sufficiently small  $u$  such that  $dCu - 1/(\sqrt{2}Cv'(t))$  is negative for all  $t$ . Since  $x$  is bounded, it further holds for all sufficiently small  $u$  that the  $\mathcal{O}(u^3 x^3(t))$  term does not affect the sign: With such a choice of  $u$ , it holds that  $d \tanh(Cxu) + a$  is negative for all  $t$ , and hence

$$y(\epsilon, t) \leq (d \tanh(Cx(t)\epsilon) + a(t)) \sinh(Cx(t)\epsilon) < 0$$

for all values of  $t$  and all sufficiently small  $\epsilon > 0$ . Since

$$y(\epsilon, t) = g(\dot{\alpha}_\epsilon(t), \dot{\alpha}_\epsilon(t))$$

this shows that the curve  $\alpha_\epsilon$  is timelike.

**Claim VIII: For all sufficiently small  $\epsilon > 0$ , the curve  $\alpha_\epsilon$  has infinite  $g'$ -length**

For each (negative) integer  $k$ , let  $L_k(u)$  be the  $g'$ -length of the restriction of  $\alpha_u$  to  $[k, k+1]$ . By the formula for the first variation of arc length ([28, Proposition 10.2])

$$\begin{aligned} L'_k(0) &= - \int_k^{k+1} g'(\nabla_{\dot{\alpha}_0} \dot{\alpha}_0, V) dt + g'(\dot{\alpha}_0, V)|_k^{k+1} \\ &= - \int_k^{k+1} f(t) g'(\dot{\alpha}_0, V) dt + g'(\dot{\alpha}_0, V)|_k^{k+1} \\ &= - \int_k^{k+1} \frac{f(t)}{\sqrt{2}} dt. \end{aligned}$$

We here used that  $g'(\alpha_0, V) = 1/\sqrt{2}$  by definition of  $g'$  and  $\alpha_0$ . Since  $f$  is bounded, we know that  $L'_k(0)$  is bounded uniformly in  $k$ . This means that for all sufficiently small  $\epsilon > 0$  it holds that  $L_k(\epsilon) > 1/2$  for all  $k$ . (Recall that  $L_k(0) = 1$  since  $\alpha_0$  is parameterized by arc length.) This means that the length of  $\alpha_\epsilon$  is

$$\sum_{k < 0} L_k(\epsilon) \geq \sum_{k < 0} 1/2 = \infty.$$

**Claim IX:** For all sufficiently small  $\epsilon > 0$ , the curve  $\alpha_\epsilon$  belongs to the interior of  $D^+(S_0)$

Since the curve  $\alpha_\epsilon$  for  $\epsilon > 0$  is a variation to the past of  $\alpha_0$ , it belongs to the open set  $I^-(H^+(S_0))$ . We will first show that  $I^-(H^+(S_0)) \cap I^+(S_0) \subseteq D^+(S_0)$ , and then show that  $\alpha_\epsilon$  belongs to  $I^-(H^+(S_0)) \cap I^+(S_0)$  for all sufficiently small  $\epsilon > 0$ . We will then have shown that  $\alpha_\epsilon$  belongs to an open set contained in  $D^+(S_0)$ , and hence it must belong to the interior of  $D^+(S_0)$ .

Let  $p \in I^-(H^+(S_0)) \cap I^+(S_0)$ . We will first show that  $p \in \overline{D^+(S_0)}$ , and then that  $p \in D^+(S_0)$ . That  $p \in I^-(H^+(S_0))$  means that there is some future-directed timelike curve  $\lambda$  from  $p$  to  $H^+(S_0)$ . This curve cannot pass  $S_0$ , since  $p$  lies to the future of  $S_0$  and  $S_0$  is achronal. Suppose now that  $\kappa$  is a future-directed past inextendible timelike curve with future endpoint  $p$ . By concatenating  $\kappa$  and  $\lambda$  and smoothing (in a neighborhood of  $p$  which is disjoint from  $S_0$ , which exists since  $p \in I^+(S_0)$ ) we obtain a past-inextendible timelike curve with future endpoint in  $H^+(S_0)$ . Since  $H^+(S_0) \subseteq \overline{D^+(S_0)}$ , this combined curve must intersect  $S_0$ . Since the curve  $\lambda$  does not intersect  $S_0$ , the curve  $\kappa$  must do so. This proves that every past-inextendible timelike curve  $\kappa$  through  $p$  must intersect  $S_0$ , so that  $p \in \overline{D^+(S_0)}$ . By the same argument, all points in the interior of  $\lambda$  belong to  $\overline{D^+(S_0)}$ . Let  $q$  be some point in the interior of  $\lambda$ . Since  $\lambda$  is timelike,  $q \in I^+(p)$ . Since  $I^+(p)$  is open, it is a neighborhood of  $q$ . Since  $q \in \overline{D^+(S_0)}$  it is a limit point of  $D^+(S_0)$ . This means that the neighborhood  $I^+(p)$  of  $q$  must contain some point  $r \in D^+(S_0) \cap I^+(p)$ . Let  $\hat{\lambda}$  be a future-directed timelike curve from  $p$  to  $r$ . Now let  $\kappa$  be a future-directed past inextendible causal curve with future endpoint  $p$ . Concatenating  $\kappa$  with  $\hat{\lambda}$  and smoothing (again in a neighborhood of  $p$  which is disjoint from  $S_0$ ) we obtain a past-inextendible causal curve with future endpoint  $r$ . Since  $r \in D^+(S_0)$ , this curve must intersect  $S_0$ . Since  $r \in I^+(p) \subseteq I^+(S_0)$  and  $S_0$  is achronal, the curve  $\hat{\lambda}$  cannot intersect  $S_0$ . This means that  $\kappa$  must intersect  $S_0$ . This proves that every past-inextendible causal curve  $\kappa$  through  $p$  must intersect  $S_0$ , so that  $p \in D^+(S_0)$ .

Since  $\alpha_0$  is a curve in  $H^+(S_0)$  and  $\alpha_\epsilon$  is a variation to the past for  $\epsilon > 0$  we know that  $\alpha_\epsilon$  belongs to  $I^-(H^+(S_0))$ . Since  $g'(V, V) = 1$  and  $x$  is bounded by  $|2\nu_0|$ , equation (B.1) implies that the distance from a point on  $\alpha_\epsilon$  to  $\alpha_0$  cannot exceed  $|2\nu_0\epsilon|$ . Since  $H^+(S_0) \cap K$  is compact and disjoint from the closed set  $S_0$ , the  $g'$ -distance from  $H^+(S_0)$  to  $S_0$  is positive. Choosing  $\epsilon > 0$  so small that  $|2\nu_0\epsilon|$  is smaller than this distance, we know that  $\alpha_\epsilon$  does not intersect  $S_0$ . To see that  $\alpha_\epsilon(t) \in I^+(S_0)$  for some  $t$ , note that no curve  $\alpha_u$  with  $0 \leq u \leq \epsilon$  can intersect  $S_0$  so that the timelike curve  $\lambda: [-\epsilon, 0] \rightarrow M$  defined by  $\lambda(u) = \alpha_{-u}(t)$  does not intersect  $S_0$ . Extend  $\lambda$  to some past inextendible timelike curve. Then  $\lambda$  is a past inextendible timelike curve with future endpoint  $\lambda(0) = \alpha_0(t) \in H^+(S_0)$ , so  $\lambda$  must intersect

$S_0$ . Since  $\lambda$  passes through  $\alpha_\epsilon(t)$ , we know that  $\alpha_\epsilon(t) \in I^+(S_0)$ . Since  $t$  was arbitrary, we have now shown that the image of  $\alpha_\epsilon$  belongs to  $I^-(H^+(S_0)) \cap I^+(S_0)$  for all sufficiently small  $\epsilon > 0$ . As noted previously, this together with the fact that  $I^-(H^+(S_0)) \cap I^+(S_0) \subseteq D^+(S_0)$  shows that the image of  $\alpha_\epsilon$  belongs to the interior of  $D^+(S_0)$ .

**Proof part X(Contradiction ensues):** We have now shown that if we choose  $\epsilon > 0$  small enough, then  $\alpha_\epsilon$  is a timelike curve of infinite  $g'$ -length, contained in the interior of  $D^+(S_0)$ . Since it has infinite  $g'$ -length and belongs to the compact set  $\overline{W}$ , it must have a limit point. Since it is timelike, the existence of this limit point implies that the strong causality condition cannot hold in any open neighborhood of  $\alpha_\epsilon$ . However, the interior of  $D^+(S_0)$  is an open neighborhood of  $\alpha_\epsilon$  satisfying the strong causality condition (by Proposition A.2.21), so we have arrived at a contradiction. Hence  $\gamma$  cannot be incomplete in the past direction.  $\square$

## B.2 Geodesically spanned null hypersurfaces

The following is a version of [28, Lemma 8.6] for geodesics normal to a submanifold. We also note which regularity is necessary for the result to hold.

**Proposition B.2.1.** *Let  $(M, g)$  be a spacetime of dimension  $n+1$  and let  $N \subset M$  be a  $C^2$  submanifold of codimension 2. Let  $\mathbf{n}$  denote a  $C^1$  normal null vector field along  $N$ . Consider the normal exponential map  $\exp: \mathbb{R} \times N \rightarrow M$  defined by*

$$\exp(t, p) = \exp_p(t\mathbf{n}_p)$$

*where  $\exp_p$  is the exponential map at the point  $p$ . Suppose that  $\mathcal{O} \subset \mathbb{R} \times N$  is an open subset such that  $\mathcal{H} := \exp(\mathcal{O})$  is an embedded  $C^1$  hypersurface in  $M$  and the pushforward  $\exp_*$  is injective on  $\mathcal{O}$ . Then  $\mathcal{H}$  is a null hypersurface.*

*Proof.* Choose a point  $q = \exp(t, p) \in \mathcal{H}$  and let  $\gamma$  denote the null geodesic  $s \mapsto \exp(s, p)$ . Our goal is to show that every vector  $W \in T_q\mathcal{H}$  is orthogonal to  $\dot{\gamma}(t)$ , thereby proving that  $T_q\mathcal{H}$  is a null hyperplane.

Since  $\exp_*$  is injective at  $\exp(t, p)$ , it is also surjective for dimensional reasons. This means that  $W$  has some preimage in  $T_{(t,p)}(\mathbb{R} \times N)$ . Denote this preimage by  $(\zeta, Z)$ , where we make use of the canonical isomorphism  $T_{(t,p)}(\mathbb{R} \times N) \cong T_t\mathbb{R} \times T_pN$ . The pushforward is linear so  $\exp_*(\zeta, Z) = \exp_*(\zeta, 0) + \exp_*(0, Z)$ . Note that  $\exp_*(\zeta, 0)$  is tangent to  $\gamma$ , so  $g(\exp_*(\zeta, 0), \dot{\gamma}(t)) = 0$ . Hence

$$g(W, \dot{\gamma}(t)) = g(\exp_*(\zeta, 0) + \exp_*(0, Z), \dot{\gamma}(t)) = g(\exp_*(0, Z), \dot{\gamma}(t)).$$

Let  $\alpha: (-1, 1) \rightarrow N$  be a curve with  $\alpha(0) = p$  and  $\dot{\alpha}(0) = Z$ . Consider the two-parameter map

$$\mathbf{x}(s, u) = \exp(st, \alpha(u))$$

defined for  $s \in [0, 1]$  and  $u \in (0, 1)$ . Let  $V$  be a vector field along  $\gamma$  defined by

$$V(s) = \mathbf{x}_u(s, 0).$$

Each curve  $s \mapsto \mathbf{x}(s, u)$  is a geodesic, so the map  $\mathbf{x}$  is a variation through geodesics. Hence  $V$  is a Jacobi vector field. The curve  $u \mapsto \mathbf{x}(0, u)$  is contained in  $N$  so  $V(0)$  is tangent to  $N$ . By assumption on  $\mathbf{n}$ , the vector  $\dot{\gamma}(0)$  is orthogonal to  $N$ , so

$$g(V(0), \dot{\gamma}(0)) = 0.$$

Let  $T$  denote the vector field  $\mathbf{x}_s$  along the map  $\mathbf{x}$ . Partial derivatives of two-parameter maps commute by [28, Proposition 4.44] so

$$V'(0) = \mathbf{x}_{us}(0, 0) = \mathbf{x}_{su}(0, 0) = \nabla_Z T.$$

Hence

$$g(V'(0), T) = g(\nabla_Z T, T) = \frac{1}{2} Z g(T, T) = 0$$

since  $T$  is tangent to null curves. Since  $\mathbf{x}_s(0, 0) = \dot{\gamma}(0)$  we have shown that

$$g(V'(0), \dot{\gamma}(0)) = 0.$$

By [28, Lemma 8.7], the fact that  $V(0)$  and  $V'(0)$  are both orthogonal to the geodesic  $\gamma$ , together with the fact that  $V$  is a Jacobi field along  $\gamma$ , implies that  $V(s)$  is orthogonal to  $\gamma$  for all  $s$ . In particular,

$$g(V(1), \dot{\gamma}(t)) = 0.$$

Computing  $V(1)$  we see that

$$V(1) = \mathbf{x}_u(1, 0) = \exp_*(0, \dot{\alpha}(0)) = \exp_*(0, Z).$$

Hence

$$g(W, \dot{\gamma}(t)) = 0$$

for all  $W \in T_q \mathcal{H}$ . Since  $q = \exp(t, p)$  was arbitrary, this shows that each tangent plane of  $\mathcal{H}$  is a null hyperplane, so that  $\mathcal{H}$  is a null hypersurface.  $\square$

### B.3 Generalized Jet Transversality

The notation for jet bundles used in this section is defined in Section A.5. Recall in particular that each jet bundle has an associated source map and target map. We will call these maps  $\pi^s$  and  $\pi^t$ , respectively.

We need a version of the Thom Transversality Theorem which applies to sections of a bundle. To prove this adapted theorem, we need the following lemma, which can be found as [17, Lemma 4.6, Chapter II], or as a special case of [20, Theorem 2.7, Chapter 3].

**Lemma B.3.1.** *Let  $X, B$  and  $Z$  be smooth manifolds with  $V$  a submanifold of  $Z$ . Let  $\rho: B \rightarrow C^\infty(X, Y)$  be a mapping (not necessarily continuous) and define  $\Phi: X \times B \rightarrow Y$  by  $\Phi(x, b) = \rho(b)(x)$ . Assume that  $\Phi$  is smooth and transverse to  $V$ . Then the set  $\{b \in B \mid \rho(b) \pitchfork V\}$  is dense in  $B$ .*

We will also need the following straightforward generalization of Proposition 4.5 in Chapter II of [17].

**Proposition B.3.2.** *Let  $X$  and  $Y$  be smooth manifolds and let  $W$  be a submanifold of  $Y$ . Let  $C$  be a compact subset of  $W$ . Let*

$$T_C = \{f \in C^\infty(X, Y) \mid f \pitchfork W \text{ on } C\}.$$

*Then  $T_C$  is an open subset of  $C^\infty(X, Y)$  in the Whitney  $C^1$  topology.*

*Proof.* For any subset  $V \subseteq J^1(X, Y)$ , let

$$M(V) = \{f \in C^\infty(X, Y) \mid j^1 f(X) \subseteq V\}.$$

Recall that the sets  $M(V)$  where  $V \subseteq J^1(X, Y)$  is open form a basis of the Whitney  $C^1$  topology on  $C^\infty(X, Y)$ .

Note that if  $\sigma \in J^1(X, Y)$  has source  $x$ , then the tangent map  $(f_*)_x: T_x X \rightarrow T_{f(x)} Y$  is independent of the choice of representative  $f$  of  $\sigma$ . This means that it makes sense to speak of a 1-jet (rather than a representative of it) being transverse to  $W$  at its source. Define the set  $\Omega \subseteq J^1(X, Y)$  by

$$\Omega = \{\sigma \in J^1(X, Y) \mid \pi^t(\sigma) \notin C \text{ or } \sigma \text{ is transverse to } W \text{ at } \pi^s(\sigma)\}.$$

Note that  $f \pitchfork_p W$  on  $C$  if and only if  $j^1 f(X) \subseteq \Omega$ . Hence  $T_C = M(\Omega)$ , so  $T_C$  is open if  $\Omega$  is open. We will show that  $J^1(X, Y) \setminus \Omega$  is closed. By definition of  $\Omega$

$$J^1(X, Y) \setminus \Omega = \{\sigma \in J^1(X, Y) \mid \pi^t(\sigma) \in C \text{ and } \sigma \text{ is **not** transverse to } W \text{ at } \pi^s(\sigma)\}.$$

Consider a sequence  $(\sigma_i)_{i=1}^\infty$  in  $J^1(X, Y) \setminus \Omega$ , converging to some  $\sigma \in J^1(X, Y)$ . Since the target  $\pi^t(\sigma_i)$  of each  $\sigma_i$  is in  $C$  and  $C$  is a closed set,  $\pi^t(\sigma) \in C$  as well. The goal is now to show that  $\sigma$  is not transverse to  $W$  at  $\pi^s(\sigma)$ , for then  $\sigma \in J^1(X, Y) \setminus \Omega$  and we are done. Choose closed coordinate neighborhoods  $U_X$  and  $U_Y$  of  $\pi^s(\sigma)$  and  $\pi^t(\sigma)$ , respectively, with surjective charts  $\psi_X: U_X \rightarrow \mathbb{D}^n$  and  $\psi_Y: U_Y \rightarrow \mathbb{D}^m$ . Here  $\mathbb{D}^k$  denotes the closed unit  $k$ -disk. We may choose  $\psi_Y$  so that  $\psi_Y(W \cap U_Y) = \mathbb{D}^m \cap \mathbb{R}^k$  where  $\mathbb{R}^k \subseteq \mathbb{R}^m$  denotes a fixed choice of  $k$ -dimensional subspace. Let  $J = \{\tau \in J^1(X, Y) \mid \pi^s(\tau) \in U_X \text{ and } \pi^t(\tau) \in U_Y\}$ . Since  $U_X$  and  $U_Y$  are neighborhoods of  $\pi^s(\sigma)$  and  $\pi^t(\sigma)$ , all but finitely many of the  $\sigma_i$  belong to  $J$ . Each  $\tau \in J$  determines

a linear map from  $\mathbb{R}^n$  to  $\mathbb{R}^m/\mathbb{R}^k$  in the following way. Let  $\pi: \mathbb{R}^m \rightarrow \mathbb{R}^m/\mathbb{R}^k$  denote the projection. Define  $\eta: J \rightarrow \text{Hom}(\mathbb{R}^n, \mathbb{R}^m/\mathbb{R}^k)$  by

$$\eta([g]) = (\pi \circ \psi_Y \circ g \circ \psi_X^{-1})_*.$$

This definition is independent of the particular representative  $g$  of  $[g] \in J$ , since it depends only on the tangent map of  $g$ . Note that  $\eta([g])$  can be thought of as the coordinate representation of the tangent map of  $g$ , composed with the linear projection onto the tangent space of  $W$ . The nice thing about the map  $\eta$  is that  $\eta(\tau)$  is surjective if and only if  $\tau$  is transverse to  $W$  at  $\pi^s(\tau)$ . By noting that  $\eta([g]) = (\pi \circ \psi_Y)_* \circ g_* \circ (\psi_X^{-1})_*$  we see that  $\eta([g])$  depends continuously on  $g_*$ , which in turn depends continuously on  $[g]$ , so  $\eta$  is continuous. Note now that the set

$$F = \{A \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m/\mathbb{R}^k) \mid A \text{ is not surjective}\}$$

is (after a choice of bases) the zero set of an algebraic equation involving determinants, and as such closed. This means that  $\eta^{-1}(F)$  is also closed. Since  $\sigma_i \in \eta^{-1}(F)$  for all  $i$  (except possibly for the finitely many which do not belong to  $J$ ) by assumption, it holds that  $\sigma \in \eta^{-1}(F)$ . This means that  $\eta(\sigma)$  is not surjective, so that  $\sigma$  is not transverse to  $W$  at  $\pi^s(\sigma)$ . This means that  $\sigma \in J^1(X, Y) \setminus \Omega$ , and so we have shown that  $\Omega$  is open. As noted in the beginning of the proof, this means that  $T_C$  is open, and so we are done.  $\square$

We now state the transversality theorem we need.

**Proposition B.3.3.** *Let  $L \rightarrow M$  and  $K \rightarrow M$  be a smooth fiber bundles over a smooth manifold. Suppose that  $W$  is a submanifold of the product bundle  $J^r L \oplus K$ . Endow  $\Gamma^\infty(L)$  with the Whitney  $C^{r+1}$  topology and let*

$$T_W = \{\phi \in \Gamma^\infty(L) \mid (j^r \phi \circ \pi_M, \text{id}_K) \pitchfork W\}$$

where  $\pi_M: K \rightarrow M$  denotes the projection. Then  $T_W$  is a residual subset of  $\Gamma^\infty(L)$ .

*Proof.* Our proof mimics the proof of the Thom Transversality Theorem given in [17, Theorem 4.9, Chapter II]. We begin by introducing some notation. The spaces  $L$ ,  $K$ ,  $J^r L$  and all fiber bundle products of them are fiber bundles over  $M$ . We will denote the fiber bundle projection maps from each of these bundles onto  $M$  by a single symbol  $\pi_M$ , relying on context to infer which map is intended. (Note that  $\pi_M: J^r L \rightarrow M$  agrees with the source map  $\pi^s: J^r L \rightarrow M$ . We shall not use the latter notation in the present proof.) We extend the target map  $\pi^t: J^r L \rightarrow L$  to a map  $\tilde{\pi}^t: J^r L \oplus K \rightarrow L \oplus K$  by letting it be the identity in the second component:  $\tilde{\pi}^t(\sigma, k) = (\pi^t(\sigma), k)$ .



**Proof part I (Localization):** The goal is to express  $T_W$  as a countable intersection of open dense subsets of  $\Gamma^\infty(L)$ . To do this, we begin by covering  $W$  by countably many open subsets  $W_1, W_2, \dots$  such that each  $W_\gamma$  satisfies that

- the closure  $\overline{W_\gamma}$  of  $W_\gamma$  in  $J^r L \oplus K$  is contained in  $W$ ,
- $\overline{W_\gamma}$  is compact,
- there is (where  $F_L$  and  $F_K$  denote the fibers of  $L$  and  $K$ ) an open trivializing coordinate neighborhood  $M_\gamma \times L_\gamma \times K_\gamma \subseteq M \times F_L \times F_K$  of the image of  $\overline{W_\gamma}$  under the map  $\tilde{\pi}^t := (\pi^t, \text{id}_K): J^r L \oplus K \rightarrow L \oplus K$ ,
- $\overline{M_\gamma}$  is compact.

To see that it is possible to construct such sets, note that it is sufficient to construct such a neighborhood of each point, since by second countability of  $W$  every open cover of  $W$  has a countable subcover. To choose a neighborhood  $W_p$  of a point  $p$ , begin by choosing an open trivializing coordinate neighborhood  $M_p \times L_p \times K_p \subseteq M \times F_L \times F_K$  of  $(\pi^t, \text{id}_K)(p)$ . Shrinking  $M_p$  if necessary, we may make  $\overline{M_p}$  compact. The inverse image of  $M_p \times L_p \times K_p$  under  $(\pi^t, \text{id}_K)$  is an open neighborhood  $V$  of  $p$ , and  $V \cap W$  is open in  $W$  since  $W$  is an embedded submanifold. We may then choose  $W_p \subseteq W$  to be an open neighborhood of  $p$  such that  $\overline{W_p}$  is compact and contained in  $V \cap W$ , since  $W$  is an embedded submanifold. This completes the construction.

For each  $\gamma$ , let

$$T_\gamma = \{\phi \in \Gamma^\infty(L) \mid (j^r \phi \circ \pi_M, \text{id}_K) \pitchfork W \text{ on } \overline{W_\gamma}\}.$$

If we can show that each  $T_\gamma$  is open and dense, then  $T_W$  is residual since

$$T_W = \bigcap_\gamma T_\gamma.$$

Fix  $\gamma$  for the remainder of the proof. Showing that  $T_\gamma$  is open can be done by an application of Proposition B.3.2. We do this first, and then devote the rest of the proof to showing that  $T_\gamma$  is dense.

**Claim II:  $T_\gamma$  is open**

We begin by showing that  $T_\gamma$  is open. Let

$$T = \{f \in C^\infty(K, J^r L \oplus K) \mid f \pitchfork W \text{ on } \overline{W_\gamma}\}.$$

Then  $T_\gamma$  is the inverse image of  $T$  under the map  $\Gamma^\infty(L) \rightarrow C^\infty(M, J^r L)$  defined by  $\phi \mapsto (j^r \phi \circ \pi_M, \text{id}_K)$ . This map is continuous when  $\Gamma^\infty(L)$  is given the Whitney  $C^{r+1}$  topology and  $C^\infty(M, J^r L)$  is given the Whitney  $C^1$  topology. It is then

sufficient to show that  $T$  is open in the Whitney  $C^1$  topology on  $C^\infty(M, J^r L)$ . By Proposition B.3.2 (taking  $X = K$  and  $Y = J^r L \oplus K$  and  $C = \overline{W_\gamma}$ ) this is indeed the case.

**Proof part III(Construction of local polynomial perturbations):** We will now show that  $T_\gamma$  is dense in  $\Gamma^\infty(L)$ . The strategy we employ involves local polynomial perturbations of sections of  $L$ . We begin by describing the construction of such polynomial perturbations. They will be defined with respect to

- a choice of coordinate chart  $\psi_M: M_\gamma \rightarrow \mathbb{R}^n$ ,
- a choice of coordinate chart  $\psi_L: L_\gamma \rightarrow \mathbb{R}^m$ ,
- a choice of smooth cut-off function  $\zeta_M: M \rightarrow [0, 1]$  such that
  - $\zeta_M(x) = 1$  for all  $x$  in some neighborhood of  $\pi_M(\overline{W_\gamma})$ ,
  - $\zeta_M(x) = 0$  if  $x \notin M_\gamma$ ,
- a choice of smooth cut-off function  $\zeta_L: \mathbb{R}^m \rightarrow [0, 1]$  such that
  - $\zeta_L(x) = 1$  for all  $x$  in some neighborhood of  $\psi_L(p_L(\tilde{\pi}^t(\overline{W_\gamma})))$ , where  $p_L: M_\gamma \times L_\gamma \times K_\gamma \rightarrow L_\gamma$  denotes the projection,
  - $\zeta_L(x) = 0$  if  $x \notin \psi_L(L_\gamma)$ .

Fix such functions  $\psi_M$ ,  $\psi_L$ ,  $\zeta_M$  and  $\zeta_L$ . Note that a choice of cut-off function  $\zeta_M$  exists since  $\pi_M(\overline{W_\gamma})$  is compact and contained in the open set  $M_\gamma$ , and similarly for  $\zeta_L$ . A section  $\sigma \in \Gamma^\infty(L)$  can locally on the trivializing coordinate neighborhood  $M_\gamma$  be viewed as a function  $f: M_\gamma \rightarrow F_L$ . We would like to perturb it by a polynomial function  $p: \mathbb{R}^n \rightarrow \mathbb{R}^m$ . The straightforward way of doing this in coordinates is forming  $\theta: x \mapsto \psi_L(f(x)) + p(\psi_M(x))$ . However, there is no guarantee that the image of this ends up in the image of  $\psi_L$ , so we will not be able to lift this function back to the manifold. Moreover, we would have perturbed  $f$  close to the boundary of  $M_\gamma$  and so would not be able to patch the perturbed function together with the original section. This is where the cut-off function come in. We define *the perturbation of  $f: M_\gamma \rightarrow F_L$  by a polynomial function  $p: \mathbb{R}^n \rightarrow \mathbb{R}^m$  (with respect to  $\psi_M$ ,  $\psi_L$ ,  $\zeta_M$  and  $\zeta_L$ )* to be the function  $f_p: M_\gamma \rightarrow F_L$  defined by

$$f_p(x) = \begin{cases} f(x) & \text{if } f(x) \notin L_\gamma, \\ \psi_L^{-1}(\psi_L(f(x)) + \tau(x)p(\psi_M(x))) & \text{otherwise,} \end{cases}$$

where the auxiliary function  $\tau$  is defined by

$$\tau(x) = \zeta_M(x)\zeta_L(\theta(x)),$$

$$\theta(x) = \psi_L(f(x)) + p(\psi_M(x)).$$

The idea is that  $f_p(x)$  is a convex combination of  $f(x)$  and what by abuse of notation can be written  $f(x) + p(x)$ , where the interpolation parameter is determined by the cut-off functions. The function  $f_p$  has a couple of notable features:

- $f_p$  and  $f$  agree smoothly at the boundary of  $M_\gamma$ , in the sense that they have the same smooth extensions to neighborhoods of  $M_\gamma$ .
- Suppose that  $p$  is a polynomial. Then there is some neighborhood of the set  $\pi_M(\overline{W_\gamma}) \times p_L(\pi^t(\overline{W_\gamma}))$  such that if  $(x, f(x))$  belongs to this neighborhood and  $p(\psi_M(x))$  is sufficiently small then the coordinate representation of  $f_p$  is precisely  $\psi_L(f(x)) + \tau(x)p(\psi_M(x))$ . In particular, for each jet  $v \in J^r(M_\gamma, F_L)$  with source  $x \in \pi_M(\overline{W_\gamma})$  and target  $f(x) \in p_L(\pi^t(\overline{W_\gamma}))$  there is a polynomial (of degree at most  $r$ ) such that  $j^r f(x) = v$ .

The first observation means that we may extend this construction to a local perturbation of a section, rather than of a local section. Let  $\sigma \in \Gamma^\infty(L)$  have the local representation  $f: M_\gamma \rightarrow F_L$  in the trivializing neighborhood  $M_\gamma$ . Let  $i: M_\gamma \times F_L \rightarrow L$  denote the local trivialization. Define *the perturbation of  $\sigma$  by a polynomial function  $p: \mathbb{R}^n \rightarrow \mathbb{R}^m$  (with respect to  $\psi_M, \psi_L, \zeta_M$  and  $\zeta_L$ )* to be

$$\sigma_p(x) = \begin{cases} \sigma(x) & \text{if } x \notin M_\gamma, \\ i(x, f_p(x)) & \text{otherwise.} \end{cases}$$

Since  $f$  and  $f_p$  agree smoothly at the boundary of  $M_\gamma$  (by the use of the cut-off function  $\zeta_M$  in the definition of  $f_p$ ) this is a smooth section. By the second observation about  $f_p$  it holds that any for a fixed section  $\sigma \in \Gamma^\infty(L)$ , any jet  $v \in J^r L$  with source  $x \in \pi_M(\overline{W_\gamma})$  and target  $\sigma(x) \in p_L(\pi^t(\overline{W_\gamma}))$  may be realized as  $j^r \sigma_p$  for some polynomial  $p$  of degree at most  $r$ . This is the crucial observation; a finite-dimensional family of perturbations are sufficient to realize any  $r$ -jet.

**Proof part IV(Application of Lemma B.3.1):** We return to our goal of showing that  $T_\gamma$  is dense in  $\Gamma^\infty(L)$ . To this end, we fix some  $\sigma \in \Gamma^\infty(L)$  and aim to show that this  $\sigma$  is a limit point of  $T_\gamma$ . Let  $P$  be the set of polynomial functions  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  of degree at most  $r$ . The set  $P$  has a conventional structure of a finite-dimensional vector space, and so has a canonical topology. Endow  $P$  with this topology. With  $P' \subseteq P$  being a subset to be described later, define  $\rho: P' \rightarrow C^\infty(K, J^r L \oplus K)$  by

$$\rho(p)(x) = (j^r \sigma_p(\pi_M(x)), x).$$

We will later show that it is possible to choose  $P' \subseteq P$  such that

- $P'$  is open,

- $0 \in P'$ ,
- each point of  $\overline{W_\gamma}$  is a regular value of the evaluation map  $\Phi: K \times P' \rightarrow J^r L \oplus K$  defined by  $\Phi(x, p) = \rho(p)(x)$ . In particular,  $\Phi$  is transverse to  $W$  on some open neighborhood of  $\overline{W_\gamma}$ .

To see that we may not simply choose  $P' = P$ , consider polynomial  $p$  which is so large that  $\sigma_q = \sigma$  for all  $q$  in a neighborhood of  $p$ . (The reason this can happen is the use of the cut-off function  $\zeta_L$  in the definition of  $\sigma_q$ .) Then  $(p, t)$  will typically not be a regular point of  $\Phi$ . Suppose for the moment that we have chosen a set  $P'$  satisfying the stated hypotheses. Then  $P'$  inherits a smooth manifold structure from  $P$ . The map  $\Phi$  is smooth since  $\pi_M$  is smooth and the partial derivatives in coordinates (in particular those up to order  $r$ ) of  $\sigma_p(m)$  depend smoothly on  $m$  and  $p$ . Let  $V$  be an open neighborhood of  $\overline{W_\gamma}$  on which  $\Phi$  is transverse to  $W$ . (We will show in the next paragraph that such a  $V$  exists.) Then  $V$  is an open submanifold of  $W$ , so  $\Phi$  being transverse to  $W$  on  $V$  is equivalent to  $\Phi$  being transverse to  $V$ . We can now apply Lemma B.3.1 to find that there is a dense subset of  $P'$  such that if  $p$  belongs to this subset then  $\rho(p) \pitchfork V$ . In particular (since  $0 \in P'$ ) there is a sequence  $(p_i)_{i=1}^\infty$  in  $P'$  converging to  $0$  such that  $\rho(p_i) \pitchfork V$  for each  $i$ . That  $\rho(p_i) \pitchfork V$  means that  $\rho(p_i) \pitchfork W$  on  $V$  which in turn means that  $\rho(p_i) \pitchfork W$  on  $\overline{W_\gamma}$ . By definition of  $T_\gamma$  this means that  $\sigma_{p_i} \in T_\gamma$  for each  $i$ . Since  $\sigma_{p_i} \rightarrow \sigma$  in the Whitney  $C^{r+1}$  topology (in fact in most reasonable topologies on  $\Gamma^\infty(L)$ ), we have shown that  $\sigma$  is a limit point of  $T_\gamma$ . Since  $\sigma$  was arbitrary, this shows that  $T_\gamma$  is dense. Once we have constructed the set  $P'$  with the desired properties, this will complete the proof.

**Claim V: There is a set  $P' \subseteq P$  with the desired properties**

Recall that we have fixed a section  $\sigma$ , which is locally represented by the function  $f: M_\gamma \rightarrow F_L$ . Choose an open neighborhood  $U$  of  $\psi_L(p_L(\tilde{\pi}^t(\overline{W_\gamma}))) \subseteq \mathbb{R}^m$  where  $\zeta_L$  is identically equal to 1. Choose a smaller open neighborhood  $V$  of  $\psi_L(p_L(\tilde{\pi}^t(\overline{W_\gamma})))$ , such that  $\overline{V} \subset U$ . Let  $\delta$  denote the distance between  $\psi_L(p_L(\tilde{\pi}^t(\overline{W_\gamma})))$  and  $U \setminus V$ , and let  $\Delta$  denote the distance between  $V$  and  $\mathbb{R}^m \setminus U$ . It is possible to choose  $U$  and  $V$  in such a way that both  $\delta$  and  $\Delta$  are positive. We make such a choice. Let  $P' \subseteq P$  be the set of polynomials  $p$  such that  $|p(x)| < \min(\delta, \Delta)$  whenever  $x \in \psi_M(M_\gamma)$ . Since  $\psi_M(M_\gamma)$  is open with compact closure,  $P'$  is an open subset of  $P$ . It also contains the zero polynomial since  $\delta, \Delta > 0$ . It remains to be shown that each point of  $\overline{W_\gamma}$  is a regular value of the map  $\Phi: K \times P' \rightarrow J^r L \oplus K$  defined by

$$\Phi(k, p) = (j^r \sigma_p(\pi_M(k)), k).$$

(Recall that we have fixed a section  $\sigma$ .) Suppose that  $\Phi(k, p)$  belongs to  $\overline{W_\gamma}$ . Then  $\pi_M(k) \in M_\gamma$ . Let  $x = \pi_M(k)$ . The section  $\sigma_p$  is represented locally by the function

$f_p: M_\gamma \rightarrow F_L$  on  $M_\gamma$ . The condition  $|p(\psi_M(x))| < \delta$  ensures that

$$\psi_L(f_p(x)) \in \psi_L(p_L(\tilde{\pi}^t(\overline{W_\gamma}))) \implies \psi_L(f(x)) \in V.$$

It holds that  $f_p(x) \in p_L(\tilde{\pi}^t(\overline{W_\gamma})) \subseteq L_\gamma$ , so  $\psi_L(f(x)) \in V$ . The condition  $|p(\psi_M(x))| < \Delta$  ensures that

$$\psi_L(f(x)) \in V \implies \psi_L(f(x)) + p(\psi_M(x)) \in U.$$

Hence

$$f_p(y) = \psi_L^{-1}(\psi_L(f(y)) + p(\psi_M(y)))$$

in a neighborhood of  $x$ . By perturbing  $p$  slightly we may perturb the partial derivatives up to order  $k$  of  $f_p$  arbitrarily. This means that  $(x, p)$  is a regular point of the (locally defined) map  $(y, q) \mapsto j^k f_q(y)$ , which in turn means that it is a regular point of the map  $(y, q) \mapsto j^k \sigma_q(y)$ . This in turn means that  $(k, p)$  is a regular point of the map  $(\ell, q) \mapsto (j^k \sigma_q(\pi_M(\ell)), \ell)$ . The latter map is the map  $\Phi$ , so since  $(k, p)$  was an arbitrary point of  $\Phi^{-1}(\overline{W_\gamma})$  we have now shown that each point of  $\overline{W_\gamma}$  is a regular value of  $\Phi$ . This proves the claim, and hence the theorem.  $\square$

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